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KINEMATICS OF RIGID BODIES IN SPACEFLIGHT

by

T. R. Kane and P. W. Likins

Technical Report No. 204

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PREFACE

The solution of problems of dynamics involving motions of rigid bodies in spaceflight necessitates extensive use of various kinematical ideas, some of which have played such a small role in the development of technology prior to the advent of spaceflight that they have nearly disappeared from the modern literature. It is the purpose of this report to present a unified, modern treatment of the kinematical ideas that the authors believe to be most useful in dealing with problems of rigid bodies in spaceflight.

CONTENTS

1.1 Simple rotation

A motion of a rigid body or reference frame B relative to a rigid body or reference frame A is called a simple rotation of B in A if there exists a line L , called an axis of rotation, whose orientation relative to both A and B remains unaltered throughout the motion. This sort of motion is important because, as will be shown in Sec. 1.3, every change in the relative orientation of A and B can be produced by means of a simple rotation of B in A .

If $\underline{a}$ is any vector fixed in A (see Fig. 1.1.1), and $\underline{b}$ is a vector fixed in B and equal to $\underline{a}$ prior to the motion of B in A , then, when B has performed a simple rotation in A , $\underline{b}$ can be expressed in terms of the vector $\underline{a}$, a unit vector $\underline{\lambda}$ parallel to L , and the radian measure θ of the angle between two lines, L_A and L_B , which are fixed in A and B , respectively, are perpendicular to L , and are parallel to each other initially. Specifically, if θ is regarded as positive when the angle between L_A and L_B is generated by a $\underline{\lambda}$-rotation of L_B relative to L_A , that is, by a rotation during which a right-handed screw fixed in B with its axis parallel to $\underline{\lambda}$ advances in the direction of $\underline{\lambda}$ when B rotates relative to A , then

$$\underline{b} = \underline{a} \cos\theta - \underline{a} \times \underline{\lambda} \sin\theta + \underline{a} \cdot \underline{\lambda\lambda}(1 - \cos\theta) \tag{1}$$

Equivalently, if a dyadic $\underline{C}$ is defined as

$$\underline{C} \triangleq \underline{U} \cos\theta - \underline{U} \times \underline{\lambda} \sin\theta + \underline{\lambda\lambda}(1 - \cos\theta) \tag{2}$$

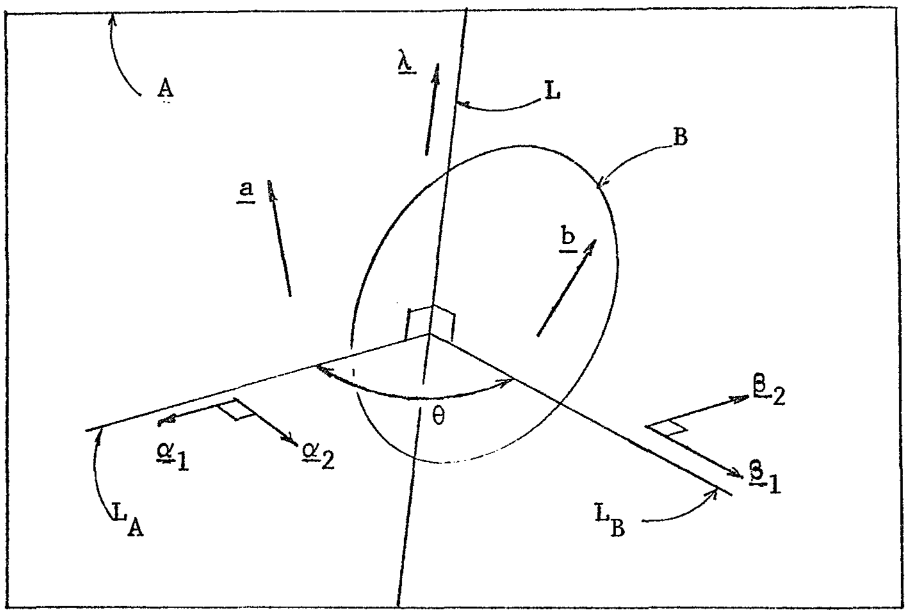

Figure 1.1.1

where $\underline{U}$ is the unit (or identity) dyadic, then

$$\underline{b} = \underline{a} \cdot \underline{C} \tag{3}$$

Derivations: Let $\underline{\alpha}_1$ and $\underline{\alpha}_2$ be unit vectors fixed in A, with $\underline{\alpha}_1$ parallel to L_A and $\underline{\alpha}_2 = \underline{\lambda} \times \underline{\alpha}_1$; and let $\underline{\beta}_1$ and $\underline{\beta}_2$ be unit vectors fixed in B, with $\underline{\beta}_1$ parallel to L_B and $\underline{\beta}_2 = \underline{\lambda} \times \underline{\beta}_1$, as shown in Fig. 1.1.1. Then, if $\underline{a}$ and $\underline{b}$ are resolved into components parallel to $\underline{\alpha}_1$, $\underline{\alpha}_2$, $\underline{\lambda}$ and $\underline{\beta}_1$, $\underline{\beta}_2$, $\underline{\lambda}$, respectively, corresponding coefficients are equal to each other because $\underline{\alpha}_1 = \underline{\beta}_1$, $\underline{\alpha}_2 = \underline{\beta}_2$, and $\underline{a} = \underline{b}$ when $\theta = 0$. In other words, $\underline{a}$ and $\underline{b}$ can be expressed as

$$\underline{a} = p\underline{\alpha}_1 + q\underline{\alpha}_2 + r\underline{\lambda} \tag{a}$$

and

$$\underline{b} = p\underline{\beta}_1 + q\underline{\beta}_2 + r\underline{\lambda} \tag{b}$$

where p, q, and r are constants.

Expressed in terms of $\underline{\alpha}_1$ and $\underline{\alpha}_2$, the unit vectors $\underline{\beta}_1$ and $\underline{\beta}_2$ are given by

$$\underline{\beta}_1 = \cos\theta\underline{\alpha}_1 + \sin\theta\underline{\alpha}_2 \tag{c}$$

and

$$\beta_2 = -\sin\,\theta\underline{\alpha}_1 + \cos\,\theta\underline{\alpha}_2 \tag{d}$$

so that, substituting into Eq. (b), one finds that

$$\underline{b} = (p\,\cos\,\theta - q\,\sin\,\theta)\underline{\alpha}_1 + (p\,\sin\,\theta + q\,\cos\,\theta)\underline{\alpha}_2 + r\underline{\lambda} \tag{e}$$

The right-hand member of Eq. (e) is precisely what one obtains by carrying out the operations indicated in the right-hand member of Eq. (1), using Eq. (a), and making use of the relationships $\underline{\lambda} \times \underline{\alpha}_1 = \underline{\alpha}_2$ and $\underline{\lambda} \times \underline{\alpha}_2 = -\underline{\alpha}_1$. Thus the validity of Eq. (1) is established; and Eq. (3) follows directly from Eqs. (1) and (2).

<u>Example</u>: A rectangular block B having the dimensions shown in Fig. 1.1.2 forms a portion of an antenna structure mounted in a spacecraft A . This block is subjected to a simple rotation in A about a diagonal of one face of B , the sense and amount of the rotation being those indicated in the sketch. The angle ϕ between the line OP in its initial and final positions is to be determined.

If $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are unit vectors fixed in A and parallel to the edges of B prior to B's rotation, then a unit vector $\underline{\lambda}$ directed as shown in the sketch can be expressed as

$$\underline{\lambda} = \frac{3\underline{a}_2 + 4\underline{a}_3}{5}$$

And, if $\underline{a}$ denotes the position vector of P relative to O prior to B's rotation, then

$$\underline{a} = -2\underline{a}_1 + 4\underline{a}_3$$

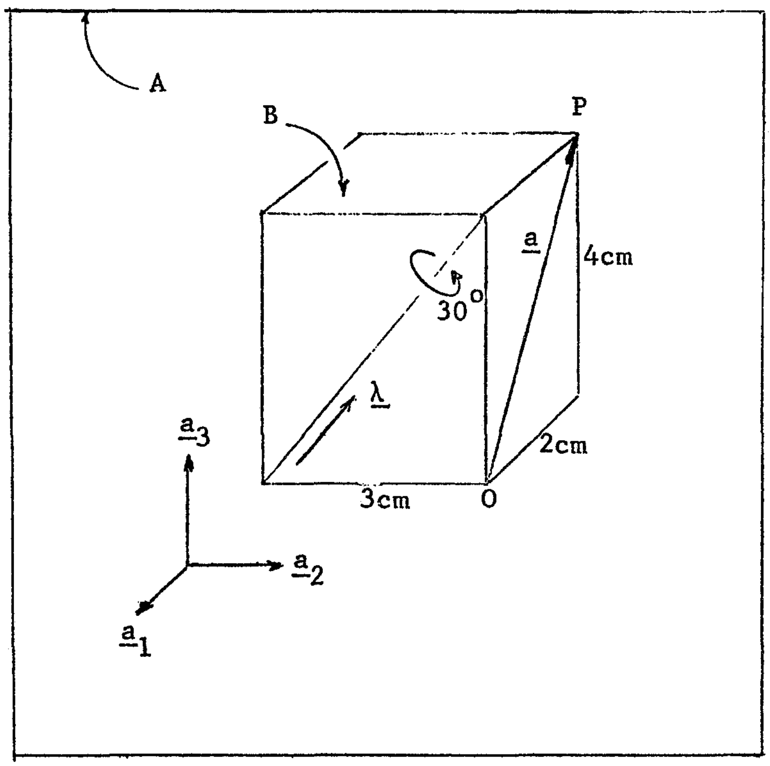
A
B
P
$\underline{a}$
4cm
30°
$\underline{\lambda}$
$\underline{a}_3$
2cm
3cm
O
$\underline{a}_2$
$\underline{a}_1$

Figure 1.1.2

$$\underline{a} \times \underline{\lambda} = \frac{-12\underline{a}_1 + 8\underline{a}_2 - 6\underline{a}_3}{5}$$

and

$$\underline{a} \cdot \underline{\lambda\lambda} = \frac{48\underline{a}_2 + 64\underline{a}_3}{25}$$

Consequently, if $\underline{b}$ is the position vector of P relative to O subsequent to B's rotation,*

$$\underline{b} \underset{(1)}{=} (-2\underline{a}_1 + 4\underline{a}_3) \cos(\pi/6) + \frac{12\underline{a}_1 - 8\underline{a}_2 + 6\underline{a}_3}{5} \sin(\pi/6)$$

$$+ \frac{48\underline{a}_2 + 64\underline{a}_3}{25} [1 - \cos(\pi/6)]$$

$$= -0.532\underline{a}_1 - 0.543\underline{a}_2 + 4.407\underline{a}_3$$

Since ϕ is the angle between $\underline{a}$ and $\underline{b}$,

$$\cos\phi = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

where $|\underline{a}|$ and $|\underline{b}|$ denote the (equal) magnitudes of $\underline{a}$ and $\underline{b}$. Hence

$$\cos\phi = \frac{(-2)(-0.532) + 4(4.407)}{(4 + 16)^{\frac{1}{2}} (4 + 16)^{\frac{1}{2}}} = 0.935$$

---

* Numbers beneath signs of equality are intended to direct attention to corresponding equations.

$$1.1$$

and

$$\phi = 20.77 \text{ deg.}$$

1.2 Direction cosines

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are two dextral sets of orthogonal unit vectors, and nine quantities C_{ij} $(i,j = 1,2,3)$, called direction cosines, are defined as

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{1}$$

then the two row matrices $[\underline{a}_1\ \underline{a}_2\ \underline{a}_3]$ and $[\underline{b}_1\ \underline{b}_2\ \underline{b}_3]$ are related to each other as follows:*

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]\ C \tag{2}$$

where C is a square matrix defined as

$$C \triangleq \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{3}$$

If a superscript T is used to denote transposition, that is, C^T is defined as

$$C^T \triangleq \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \tag{4}$$

---

* The term "matrix" is here used in an extended sense.

then Eq. (2) can be replaced with the equivalent relationship

$$[\underline{a}_1\ \underline{a}_2\ \underline{a}_3] = [\underline{b}_1\ \underline{b}_2\ \underline{b}_3]\ C^T \tag{5}$$

The matrix C, called a <u>direction</u> <u>cosine</u> <u>matrix</u>, can be employed to describe the relative orientation of two reference frames or rigid bodies A and B. In that context, it can be advantageous to replace the symbol C with the more elaborate symbole ${}^AC^B$. In view of Eqs. (2) and (5), one must then regard the interchanging of superscripts as signifying transposition; that is,

$${}^BC^A \triangleq ({}^AC^B)^T \tag{6}$$

The direction cosine matrix C plays a role in a number of useful relationships. For example, if $\underline{v}$ is any vector and ${}^Av_i$ and ${}^Bv_i$ $(i = 1,2,3)$ are defined as

$${}^Av_i = \underline{v} \cdot \underline{a}_i \qquad (i = 1,2,3) \tag{7}$$

and

$${}^Bv_i \triangleq \underline{v} \cdot \underline{b}_i \qquad (i = 1,2,3) \tag{8}$$

while ${}^Av$ and ${}^Bv$ denote the row matrices having the elements

${}^{A}v_1$, ${}^{A}v_2$, ${}^{A}v_3$ and ${}^{B}v_1$, ${}^{B}v_2$, ${}^{B}v_3$, respectively, then

$$ {}^{B}v = {}^{A}v\ C \tag{9} $$

Similarly, if $\underline{D}$ is any dyadic, and ${}^{A}D_{ij}$ and ${}^{B}D_{ij}$ $(i,j = 1,2,3)$ are defined as

$$ {}^{A}D_{ij} \triangleq \underline{a}_i \cdot \underline{D} \cdot \underline{a}_j \qquad (i,j = 1,2,3) \tag{10} $$

and

$$ {}^{B}D_{ij} \triangleq \underline{b}_i \cdot \underline{D} \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{11} $$

while ${}^{A}D$ and ${}^{B}D$ denote square matrices having ${}^{A}D_{ij}$ and ${}^{B}D_{ij}$, respectively, as the elements in the i^{th} row and j^{th} column, then

$$ {}^{B}D = C^{T}\ {}^{A}D\ C \tag{12} $$

Use of the summation convention for repeated subscripts frequently makes it possible to formulate important relationships rather concisely. For example, if δ_{ij} is defined as

$$ \delta_{ij} \triangleq 1 - \frac{1}{4}(i - j)^2[5 - (i - j)^2] \qquad (i,\ j = 1,2,3) \tag{13} $$

so that δ_{ij} is equal to unity when the subscripts have the same value, and equal to zero when the subscripts have different values, then use of the summation convention permits one to express a set of six relationships governing direction cosines as

$$C_{ik} C_{jk} = \delta_{ij} \qquad (i,j = 1,2,3) \tag{14}$$

or an equivalent set as

$$C_{ki} C_{kj} = \delta_{ij} \qquad (i,j = 1,2,3) \tag{15}$$

Alternatively, these relationships can be stated in matrix form as

$$C\,C^T = U \tag{16}$$

and

$$C^T C = U \tag{17}$$

where U denotes the unit (or identity) matrix, defined as

$$U \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

Each element of the matrix C is equal to its cofactor in the determinant of C ; and, if $|C|$ denotes this determinant, then

$$|C| = 1 \tag{19}$$

Consequently, C is an orthogonal matrix, that is, a matrix whose inverse and whose transpose are equal to each other. Moreover,

$$|C - U| = 0 \tag{20}$$

Hence unity is an eigenvalue of every direction cosine matrix. In other words, for every direction cosine matrix C there exist row matrices $[\kappa_1 \ \kappa_2 \ \kappa_3]$, called eigenvectors, which satisfy the equation

$$[\kappa_1 \ \kappa_2 \ \kappa_3] \, C = [\kappa_1 \ \kappa_2 \ \kappa_3] \tag{21}$$

Suppose now that $\underline{a}_i$ and $\underline{b}_i$ (i = 1,2,3) are fixed in reference frames or rigid bodies A and B, respectively, and that B is subjected to a simple rotation in A (see Sec. 1.1); further, that $\underline{a}_i = \underline{b}_i$ (i = 1,2,3) prior to the rotation, that $\underline{\lambda}$ and θ are defined as in Sec. 1.1, and that λ_i is defined as

$$\lambda_i \overset{\Delta}{=} \underline{\lambda} \cdot \underline{a}_i = \underline{\lambda} \cdot \underline{b}_i \qquad (i = 1,2,3) \tag{22}$$

Then the elements of C are given by

$$C_{11} = \cos\theta + \lambda_1^2(1 - \cos\theta) \qquad (23)$$

$$C_{12} = -\lambda_3 \sin\theta + \lambda_1 \lambda_2(1 - \cos\theta) \qquad (24)$$

$$C_{13} = \lambda_2 \sin\theta + \lambda_3 \lambda_1(1 - \cos\theta) \qquad (25)$$

$$C_{21} = \lambda_3 \sin\theta + \lambda_1 \lambda_2(1 - \cos\theta) \qquad (26)$$

$$C_{22} = \cos\theta + \lambda_2^2(1 - \cos\theta) \qquad (27)$$

$$C_{23} = -\lambda_1 \sin\theta + \lambda_2 \lambda_3(1 - \cos\theta) \qquad (28)$$

$$C_{31} = -\lambda_2 \sin\theta + \lambda_3 \lambda_1(1 - \cos\theta) \qquad (29)$$

$$C_{32} = \lambda_1 \sin\theta + \lambda_2 \lambda_3(1 - \cos\theta) \qquad (30)$$

$$C_{33} = \cos\theta + \lambda_3^2(1 - \cos\theta) \qquad (31)$$

Eqs. (23) - (31) can be expressed more concisely after defining ϵ_{ijk} as

$$\epsilon_{ijk} \triangleq \frac{1}{2}(i-j)(j-k)(k-i) \qquad (i,j,k = 1,2,3) \qquad (32)$$

(The quantity ϵ_{ijk} vanishes when two or three subscripts have the same value; it is equal to unity when the subscripts appear in cyclic order, that is, in the order 1,2,3, the order 2,3,1, or the order 3,1,2; and it is equal to negative unity in all other cases.) Using the summation convention, one can then replace Eqs. (24) - (31) with

$$C_{ij} = \delta_{ij} \cos\theta - \epsilon_{ijk}\lambda_k \sin\theta + \lambda_i \lambda_j (1 - \cos\theta) \qquad (i,j = 1,2,3) \tag{33}$$

Alternatively, C_{ij} can be expressed in terms of the dyadic $\underline{C}$ defined in Eq. (1.1.2):

$$C_{ij} = \underline{a}_i \cdot \underline{a}_j \cdot \underline{C} \qquad (i,j = 1,2,3) \tag{34}$$

All of these results simplify substantially when $\underline{\lambda}$ is parallel to $\underline{a}_i$, and hence to $\underline{b}_i$ $(i = 1,2,3)$. If $C_i(\theta)$ denotes C for $\underline{\lambda} = \underline{a}_i = \underline{b}_i$, then

$$C_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{35}$$

$$C_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{36}$$

$$C_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{37}$$

It was mentioned previously that unity is an eigenvalue of every direction cosine matrix. If the elements of a direction cosine matrix C are given by Eqs. (23) - (31), then the row matrix $[\lambda_1\ \lambda_2\ \lambda_3]$ is one of the eigenvectors corresponsing to the eigenvalue unity of C ; that is,

$$[\lambda_1\ \lambda_2\ \lambda_3]\ C = [\lambda_1\ \lambda_2\ \lambda_3] \tag{38}$$

Equivalently, when $\underline{C}$ is the dyadic defined in Eq. (1.1.2), then

$$\underline{\lambda} \cdot \underline{C} = \underline{\lambda} \tag{39}$$

<u>Derivations</u>: For any vector $\underline{v}$, the following is an identity:

$$\underline{v} = (\underline{a}_1 \cdot \underline{v})\underline{a}_1 + (\underline{a}_2 \cdot \underline{v})\underline{a}_2 + (\underline{a}_3 \cdot \underline{v})\underline{a}_3$$

Hence, letting $\underline{b}_1$ play the rôle of $\underline{v}$, one can express $\underline{b}_1$ as

$$\underline{b}_1 = (\underline{a}_1 \cdot \underline{b}_1)\underline{a}_1 + (a_2 \cdot \underline{b}_1)\underline{a}_2 + (\underline{a}_3 \cdot \underline{b}_1)\underline{a}_3$$

or, by using Eq. (1), as

$$\underline{b}_1 = C_{11}\,\underline{a}_1 + C_{21}\,\underline{a}_2 + C_{31}\,\underline{a}_3$$

Similarly,

$$\underline{b}_2 = C_{12}\,\underline{a}_1 + C_{22}\,\underline{a}_2 + C_{32}\,\underline{a}_3$$

and

$$\underline{b}_3 = C_{13}\,\underline{a}_1 + C_{23}\,\underline{a}_2 + C_{33}\,\underline{a}_3$$

These three equations are precisely what one obtains when forming expressions for $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ in accordance with Eq. (2) and with the rules for matrix multiplication; and a similar line of reasoning leads to Eq. (5).

To see that Eq. (9) is valid one needs only to observe that

$$^{B}v_i \underset{(8,2)}{=} \underline{v} \cdot (\underline{a}_1\,C_{1i} + \underline{a}_2\,C_{2i} + \underline{a}_3\,C_{3i})$$

$$= \underline{v} \cdot \underline{a}_1\,C_{1i} + \underline{v} \cdot \underline{a}_2\,C_{2i} + \underline{v} \cdot \underline{a}_3\,C_{3i}$$

$$\underset{(7)}{=} \; {}^{A}v_1 \, C_{1i} + {}^{A}v_2 \, C_{2i} + {}^{A}v_3 \, C_{3i}$$

and to recall the definitions of ${}^{A}v$ and ${}^{B}v$. Similarly, Eq. (12) follows from

$$
\begin{aligned}
{}^{B}D_{ij} \underset{(11,2)}{=} & \; (\underline{a}_1 \, C_{1i} + \underline{a}_2 \, C_{2i} + \underline{a}_3 \, C_{3i}) \cdot \underline{D} \cdot (\underline{a}_1 \, C_{1j} + \underline{a}_2 \, C_{2j} + \underline{a}_3 \, C_{3j}) \\
\underset{(10)}{=} & \; C_{1i} \, ({}^{A}D_{11} \, C_{1j} + {}^{A}D_{12} \, C_{2j} + {}^{A}D_{13} \, C_{3j}) \\
+ & \; C_{2i} ({}^{A}D_{21} \, C_{1j} + {}^{A}D_{22} \, C_{2j} + {}^{A}D_{23} \, C_{3j}) \\
+ & \; C_{3i} ({}^{A}D_{31} \, C_{1j} + {}^{A}D_{32} \, C_{2j} + {}^{A}D_{33} \, C_{3j})
\end{aligned}
$$

and from the definitions of ${}^{A}D$ and ${}^{B}D$.

As for Eqs. (14) and (15), these are consequences of (using the summation convention)

$$\underline{a}_i \cdot \underline{a}_j \underset{(1,5)}{=} C_{ik} \, C_{jk} \qquad (i,j = 1,2,3)$$

and

$$\underline{b}_i \cdot \underline{b}_j = C_{ki} \, C_{kj} \qquad (i,j = 1,2,3)$$

respectively, because $\underline{a}_i \cdot \underline{a}_j$ is equal to unity when $i = j$, and equal to zero when $i \neq j$, and similarly for $\underline{b}_i \cdot \underline{b}_j$; and Eqs. (16) and (17) can be seen to be equivalent to Eqs. (14) and (15), respectively, by referring to Eqs. (3), (4), and (18) when carrying out the indicated multiplications.

To verify that each element of C is equal to its cofactor in the determinant of C, note that

$$\underline{b}_1 \underset{(2)}{=} C_{11}\underline{a}_1 + C_{21}\underline{a}_2 + C_{31}\underline{a}_3$$

and

$$\begin{aligned}\underline{b}_2 \times \underline{b}_3 \underset{(2)}{=} & (C_{22}C_{33} - C_{32}C_{23})\underline{a}_1 \\ + & (C_{32}C_{13} - C_{12}C_{33})\underline{a}_2 \\ + & (C_{12}C_{23} - C_{22}C_{13})\underline{a}_3\end{aligned}$$

so that, since $\underline{b}_1 = \underline{b}_2 \times \underline{b}_3$ because $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ form a dextral set of orthogonal unit vectors,

$$C_{11} = C_{22}C_{33} - C_{32}C_{23}$$

$$C_{21} = C_{32}C_{13} - C_{12}C_{33}$$

and

$$C_{31} = C_{12}C_{23} - C_{22}C_{13}$$

Thus each element in the first column of C (see Eq. (3)) is seen to be equal to its cofactor in C ; and, using the relationships $\underline{b}_2 = \underline{b}_3 \times \underline{b}_1$ and $\underline{b}_3 = \underline{b}_1 \times \underline{b}_2$, one obtains corresponding results for the elements in the second and third columns of C . Furthermore, expanding C by cofactors of elements of the first row, and using Eq. (14) with $i = j = 1$, one arrives at Eq. (19).

The determinant of C - U can be expresses as

$$|C - U| \underset{(3)}{=} |C| + C_{11} + C_{22} + C_{33} + C_{12}C_{21} + C_{23}C_{32} + C_{31}C_{13}$$

$$- (C_{11}C_{22} + C_{22}C_{33} + C_{33}C_{11}) - 1$$

Hence, replacing C_{11} , C_{22} , and C_{33} with their respective cofactors in $|C|$, one finds that

$$|C - U| = |C| - 1 \underset{(19)}{=} 0$$

in agreement with Eq. (20); and the existence of row matrices $[\kappa_1\ \kappa_2\ \kappa_3]$ satisfying Eq. (21) is thus guaranteed.

The equality of $\underline{\lambda} \cdot \underline{a}_i$ and $\underline{\lambda} \cdot \underline{b}_i$ in Eq. (22) is a consequence of the fact that these two quantities are equal to each other prior to the rotation of B relative to A , that is, when $\underline{a}_i = \underline{b}_i$, and that neither $\underline{\lambda} \cdot \underline{a}_i$ nor $\underline{\lambda} \cdot \underline{b}_i$ changes during the rotation, since, by construction, $\underline{\lambda}$ is parallel to a line whose orientation in both A and B remains unaltered during the rotation.

With $\underline{a}$ and $\underline{b}$ replaced by $\underline{a}_j$ and $\underline{b}_j$, respectively, Eq. (1.1.3) becomes

$$\underline{b}_j = \underline{a}_j \cdot \underline{C}$$

Hence

$$C_{ij} \underset{(1)}{=} \underline{a}_i \cdot \underline{b}_j = \underline{a}_i \cdot \underline{a}_j \cdot \underline{C} \qquad (i,j = 1,2,3)$$

which is Eq. (34). Moreover, substituting for $\underline{C}$ the expression given in Eq. (1.1.2), one finds that

$$C_{ij} = \underline{a}_i \cdot \underline{a}_j \cos\theta - \underline{a}_i \cdot \underline{a}_j \times \underline{\lambda} \sin\theta + \underline{a}_i \cdot \underline{\lambda\lambda} \cdot \underline{a}_j (1 - \cos\theta) \qquad (i,j = 1,2,3)$$

and this, together with Eq. (22) leads directly to Eqs. (23) - (31), or, in view of Eq. (32), to Eq. (33).

Eq. (35) is obtained by setting $\lambda_i = 1$ and $\lambda_2 = \lambda_3 = 0$ in Eqs. (23) - (31) and then using Eq. (3). Similarly, $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0$ lead to Eq. (36), and $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ yield Eq. (37).

Finally, Eq. (39) is derived from the observation that *

$$\underline{\lambda} \cdot \underline{C} \underset{(1.1.2)}{=} \underline{\lambda} \cdot \underline{U}\cos\theta - \underline{\lambda} \cdot \underline{\lambda}\sin\theta + \underline{\lambda}^2\underline{\lambda}(1 - \cos\theta)$$

$$= \underline{\lambda}\cos\theta + 0 + \underline{\lambda}(1 - \cos\theta) = \underline{\lambda}$$

since $\underline{\lambda}$ is a unit vector, so that $\underline{\lambda}^2 \overset{\Delta}{=} \underline{\lambda} \cdot \underline{\lambda} = 1$; and the equivalence of Eqs. (38) and (39) follows from Eqs. (22) and (34).

Example: In Fig. 1.2.1, B designates a uniform rectangular block which is part of a scanning platform mounted in a spacecraft A . Initially, the edges of B are parallel to unit vectors $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ which are fixed in A , and the platform is then subjected to a simple ninety degree rotation about a diagonal of B , as indicated in the sketch. If $\underline{I}$ is the inertia dyadic of B for the mass center B^* of B , and ${}^A I_{ij}$ is defined as

$$^A I_{ij} \overset{\Delta}{=} \underline{a}_i \cdot \underline{I} \cdot \underline{a}_j \qquad (i,j = 1,2,3)$$

what is the value of ${}^A I_{ij}$ $(i,j = 1,2,3)$ subsequent to the rotation?

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* When it is necessary to refer to an equation from an earlier section, the section number is cited together with the equation number. For example, (2.3.4) refers to Eq. (4) in Sec. 2.3.

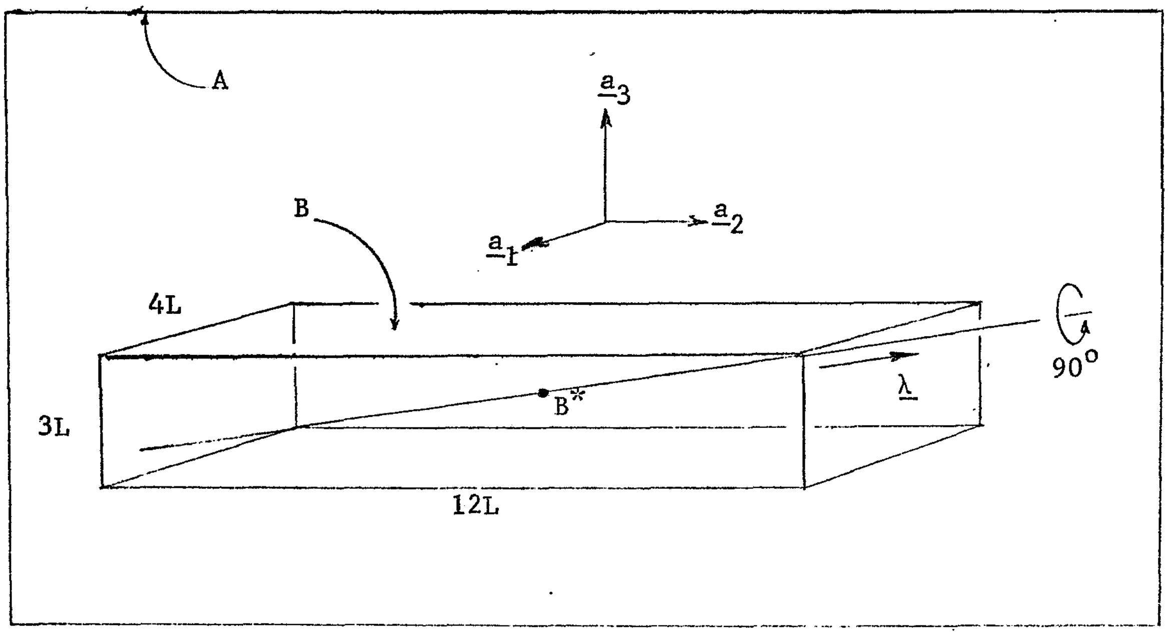
A
$\underline{a}_3$
B
$\underline{a}_1$
$\underline{a}_2$
4L
90^o
3L
B^*
$\underline{\lambda}$
12L

Figure 1.2.1

Let $\underline{b}_i$ $(i = 1,2,3)$ be a unit vector fixed in B and equal to $\underline{a}_i$ $(i = 1,2,3)$ prior to the rotation; and define ${}^B I_{ij}$ as

$${}^B I_{ij} \triangleq \underline{b}_i \cdot \underline{I} \cdot \underline{b}_j \qquad (i,j = 1,2,3)$$

Then $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are parallel to principal axes of inertia of B for B^*, so that

$${}^B I_{12} = {}^B I_{21} = {}^B I_{23} = {}^B I_{32} = {}^B I_{31} = {}^B I_{13} = 0$$

and, if m is the mass of B ,

$${}^B I_{11} = \frac{m}{12}(12^2 + 3^2)L^2 = \frac{153}{12}\, mL^2$$

$${}^B I_{22} = \frac{m}{12}(3^2 + 4^2)L^2 = \frac{25}{12}\, mL^2$$

and

$${}^B I_{33} = \frac{m}{12}(4^2 + 12^2)L^2 = \frac{160}{12}\, mL^2$$

Hence, if ${}^B I$ denotes the square matrix having ${}^B I_{ij}$ as the element in the i^{th} row and j^{th} column, then

$${}^B I = \frac{mL^2}{12} \begin{bmatrix} 153 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 60 \end{bmatrix} \qquad \text{(a)}$$

1.2

The unit vector $\underline{\lambda}$ shown in Fig. 1.2.1 can be expressed as

$$\underline{\lambda} = \frac{4\underline{a}_1 + 12\underline{a}_2 + 3\underline{a}_3}{(4^2 + 12^2 + 3^2)^{\frac{1}{2}}} = \frac{4}{13}\,\underline{a}_1 + \frac{12}{13}\,\underline{a}_2 + \frac{3}{13}\,\underline{a}_3$$

Consequently, λ_1, λ_2, and λ_3, if defined as in Eq. (22), are given by

$$\lambda_1 = \frac{4}{13}, \qquad \lambda_2 = \frac{12}{13}, \qquad \lambda_3 = \frac{3}{13}$$

and, with $\theta = \pi/2$ rad., Eqs. (23) - (31) lead to the following expression for the direction cosine matrix C:

$$\underset{(3)}{C} = \frac{1}{169}\begin{bmatrix} 16 & 9 & 168 \\ 87 & 144 & -16 \\ -144 & 88 & 9 \end{bmatrix} \tag{b}$$

If ${}^{A}I$ is now defined as the square matrix having ${}^{A}I_{ij}$ as the element in the i^{th} row and j^{th} column, then

$$\underset{(12)}{{}^{B}I = C^T\, {}^{A}I\, C} \tag{c}$$

and simultaneous pre-multiplication with C and post-multiplication with C^T gives

$$C\,{}^{B}I\,C^{T} \underset{(c)}{=} C\,C^{T}\,{}^{A}I\,C\,C^{T} \underset{(16,17)}{=} U\,{}^{A}I\,U = {}^{A}I \qquad (d)$$

Consequently,

$$^{A}I \underset{(a,b,d)}{=} \frac{mL^2}{12 \times 169 \times 169} \begin{bmatrix} 16 & 9 & 168 \\ 87 & 144 & -16 \\ -144 & 88 & 9 \end{bmatrix} \begin{bmatrix} 153 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 60 \end{bmatrix} \begin{bmatrix} 16 & 87 & -144 \\ 9 & 144 & 88 \\ 168 & -16 & 9 \end{bmatrix}$$

$$= \frac{mL^2}{12 \times 169 \times 169} \begin{bmatrix} 4557033 & -184604 & -92792 \\ -184604 & 1717417 & -1623024 \\ -92792 & -1623034 & 3379168 \end{bmatrix}$$

and

$$^{A}I_{11} = \frac{4557033\ mL^2}{12 \times 169 \times 169}$$

$$^{A}I_{12} = -\frac{184604\ mL^2}{12 \times 169 \times 169}$$

and so forth.

1.3 Euler parameters

The unit vector $\underline{\lambda}$ and the angle θ introduced in Sec. 1.1 can be used to associate a vector $\underline{\varepsilon}$, called the Euler vector, and four scalar quantities, $\varepsilon_1,\ldots,\varepsilon_4$, called Euler parameters, with a simple rotation of a rigid body B in a reference frame A by letting

$$\underline{\varepsilon} \triangleq \underline{\lambda} \sin (\theta/2) \tag{1}$$

$$\varepsilon_i \triangleq \underline{\varepsilon} \cdot \underline{a}_i = \underline{\varepsilon} \cdot \underline{b}_i \qquad (i = 1,2,3) \tag{2}$$

and

$$\varepsilon_4 \triangleq \cos (\theta/2) \tag{3}$$

where $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are dextral sets of orthogonal unit vectors fixed in A and B respectively, with $\underline{a}_i = \underline{b}_i$ $(i = 1,2,3)$ prior to the rotation. (Where a discussion involves more than two bodies or reference frames, notations such as ${}^{A}\underline{\varepsilon}^{B}$ and ${}^{A}\varepsilon_i{}^{B}$ will be used.)

The Euler parameters are not independent of each other, for the sum of their squares is necessarily equal to unity:

$$\varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2 + \varepsilon_4{}^2 = \underline{\varepsilon}^2 + \varepsilon_4{}^2 = 1 \tag{4}$$

An indication of the utility of the Euler parameters may be gleaned from the fact that the elements of the direction cosine matrix C introduced in Sec. 1.2 assume a particularly simple and orderly form when expressed in terms of $\varepsilon_1,\ldots,\varepsilon_4$: If C_{ij} is defined as

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i = 1,2,3) \tag{5}$$

then

$$C_{11} = \varepsilon_1{}^2 - \varepsilon_2{}^2 - \varepsilon_3{}^2 + \varepsilon_4{}^2 = 1 - 2\varepsilon_2{}^2 - 2\varepsilon_3{}^2 \tag{6}$$

$$C_{12} = 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) \tag{7}$$

$$C_{13} = 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4) \tag{8}$$

$$C_{21} = 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \tag{9}$$

$$C_{22} = \varepsilon_2{}^2 - \varepsilon_3{}^2 - \varepsilon_1{}^2 + \varepsilon_4{}^2 = 1 - 2\varepsilon_3{}^2 - 2\varepsilon_1{}^2 \tag{10}$$

$$C_{23} = 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \tag{11}$$

$$C_{31} = 2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4) \tag{12}$$

$$C_{32} = 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) \tag{13}$$

$$C_{33} = \varepsilon_3{}^2 - \varepsilon_1{}^2 - \varepsilon_2{}^2 + \varepsilon_4{}^2 = 1 - 2\varepsilon_1{}^2 - 2\varepsilon_2{}^2 \tag{14}$$

The Euler parameters can be repressed in terms of direction cosines in such a way that Eqs. (6) - (14) are satisfied identically. This is accomplished by taking

$$\varepsilon_1 = \frac{C_{32} - C_{23}}{4\varepsilon_4} \tag{15}$$

$$\varepsilon_2 = \frac{C_{13} - C_{31}}{4\varepsilon_4} \tag{16}$$

$$\varepsilon_3 = \frac{C_{21} - C_{12}}{4\varepsilon_4} \tag{17}$$

and

$$\varepsilon_4 = \frac{1}{2}\,(1 + C_{11} + C_{22} + C_{33})^{\frac{1}{2}} \tag{18}$$

Since Eqs. (1) and (3) are satisfied if

$$\underline{\lambda} = \frac{\varepsilon_1\underline{a}_1 + \varepsilon_2\underline{a}_2 + \varepsilon_3\underline{a}_3}{(\varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2)^{\frac{1}{2}}} \tag{19}$$

and

$$\theta = 2\cos^{-1}(\varepsilon_4) \quad , \quad 0 \leq \theta \leq \pi \tag{20}$$

one can thus find a simple rotation such that the direction cosines associated with this rotation as in Eqs. (1.2.23) - (1.2.31) are equal to corresponding elements of any direction cosine matrix C that satisfies Eq. (1.2.2). In other words, every change in the relative orientation of two rigid bodies or reference frames A and B can be produced by means of a simple rotation of B in A. This proposition is known as Euler's Theorem on rotation.

As an alternative to Eqs. (1.1.1) and (1.1.3), the relationship between a vector $\underline{a}$ fixed in a reference frame A and a vector $\underline{b}$ fixed in a rigid body B and equal to $\underline{a}$ prior to a simple rotation of B in A can be expressed in terms of $\underline{\varepsilon}$ and ε_4 as

$$\underline{b} = \underline{a} + 2[\varepsilon_4\underline{\varepsilon} \times \underline{a} + \underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a})] \tag{21}$$

<u>Derivations</u>: The equality of $\underline{\varepsilon} \cdot \underline{a}_i$ and $\underline{\varepsilon} \cdot \underline{b}_i$ [see Eq. (2)] follows from Eqs. (1) and (1.2.22); Eqs. (4) are consequences of Eqs. (1) - (3) and of the fact that $\underline{\lambda}$ is a unit vector; and Eqs. (6) - (14) can be obtained from Eqs. (1.2.23) - (1.2.31) by replacing functions of θ with functions of $\theta/2$ and using Eq. (1.2.22) together with Eqs. (1) - (4). For example,

$$C_{11} \underset{(1.2.23)}{=} 2 \cos^2 (\theta/2) - 1 + 2\lambda_1^{\,2} \sin^2 (\theta/2)$$

and

$$\lambda_1 \sin (\theta/2) \underset{(1.2.22)}{=} \underline{\lambda} \cdot \underline{a}_1 \sin (\theta/2) \underset{(1)}{=} \underline{\varepsilon} \cdot \underline{a}_1 \underset{(2)}{=} \varepsilon_1$$

while

$$\cos (\theta/2) \underset{(3)}{=} \varepsilon_4$$

Hence

$$C_{11} = 2\varepsilon_4^{\,2} - 1 + 2\varepsilon_1^{\,2} \underset{(4)}{=} \varepsilon_1^{\,2} - \varepsilon_2^{\,2} - \varepsilon_3^{\,2} + \varepsilon_4^{\,2}$$

in agreement with Eq. (1).

The validity of Eqs. (15) - (18) can be established by showing that the left-hand members of Eqs. (6) - (14) may be obtained by substituting from Eqs. (15) - (18) into the right-hand members. For example,

$$1 - 2\varepsilon_2^{\,2} - 2\varepsilon_3^{\,2}$$

$$\underset{(16\text{-}18)}{=} \frac{2(1 + C_{11} + C_{22} + C_{33}) - C_{13}^{\,2} + 2C_{13}C_{31} - C_{31}^{\,2} - C_{21}^{\,2} + 2C_{12}C_{21} - C_{12}^{\,2}}{2(1 + C_{11} + C_{22} + C_{33})}$$

$$\underset{(1.2.14)}{=} \frac{C_{11} + C_{22} + C_{33} + C_{13}C_{31} + C_{12}C_{21} + C_{11}^{\,2}}{1 + C_{11} + C_{22} + C_{33}}$$

But, since each element of C is equal to its cofactor in $|C|$,

$$C_{13}C_{31} = C_{11}C_{33} - C_{22}$$

and

$$C_{12}C_{21} = C_{11}C_{22} - C_{33}$$

Consequently

$$1 - 2\varepsilon_2^{\,2} - 2\varepsilon_3^{\,2} = \frac{C_{11} + C_{11}C_{33} + C_{11}C_{22} + C_{11}^{\,2}}{1 + C_{11} + C_{22} + C_{33}} = C_{11}$$

as required by Eq. (6).

To see that Eqs. (1) and (3) are satisfied if $\underline{\lambda}$ and θ are given by Eqs. (19) and (20), note that

$$\cos(\theta/2) \underset{(20)}{=} \varepsilon_4$$

which is Eq. (3), and that

$$\sin(\theta/2) \underset{(20)}{=} (1 - \varepsilon_4{}^2)^{\frac{1}{2}} \underset{(4)}{=} (\varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2)^{\frac{1}{2}}$$

so that

$$\underline{\lambda} \sin(\theta/2) \underset{(19)}{=} \varepsilon_1\underline{a}_1 + \varepsilon_2\underline{a}_2 + \varepsilon_3\underline{a}_3 \underset{(2)}{=} \underline{\varepsilon}$$

as required by Eq. (1). Finally, Eq. (1.1.1) is equivalent to

$$\underline{b} = \underline{a} + \underline{\lambda} \times \underline{a} \sin\theta + \underline{\lambda} \times (\underline{\lambda} \times \underline{a})(1 - \cos\theta)$$

$$\underset{(1)}{=} \underline{a} + 2\underline{\varepsilon} \times \underline{a} \cos(\theta/2) + 2\underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a})$$

$$\underset{(3)}{=} \underline{a} + 2\,[\varepsilon_4\underline{\varepsilon} \times \underline{a} + \underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a})]$$

in agreement with Eq. (21).

<u>Example</u>: Triangle ABC in Fig. 1.3.1 can be brought into the position A'B'C' by moving point A to A' without changing the orientation of the triangle and then performing a simple rotation of the triangle while keeping A fixed at A'. To find $\underline{\lambda}$, a unit vector parallel to the axis of rotation, and to determine θ, the

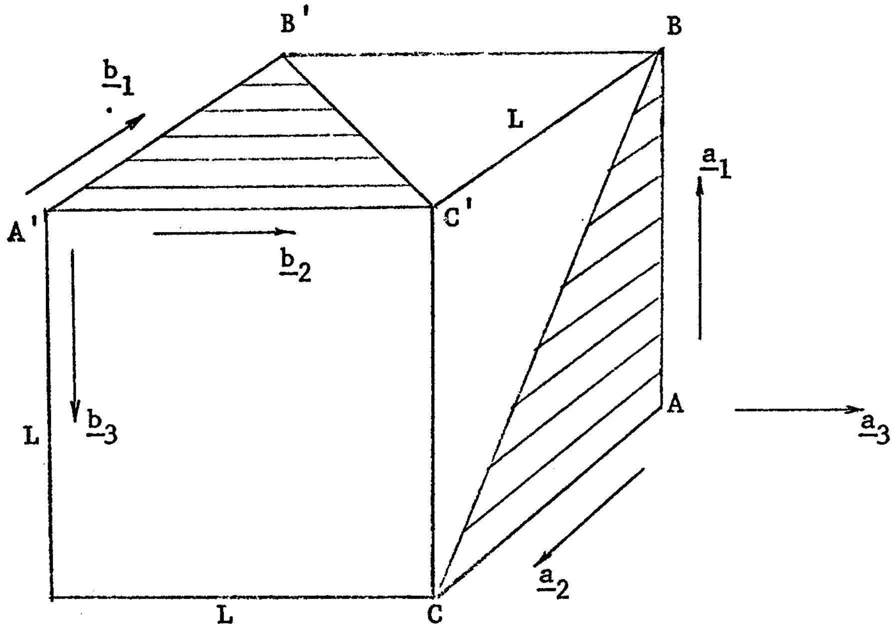
B'
B
$\underline{b}_1$
L
$\underline{a}_1$
A'
C'
$\underline{b}_2$
A
$\underline{a}_3$
L
$\underline{b}_3$
$\underline{a}_2$
L
C

Figure 1.3.1

associated angle of rotation, let the unit vectors $\underline{a}_i$ and $\underline{b}_i$ $(i = 1,2,3)$ be directed as shown in Fig. 1.3.1, thus insuring that $\underline{a}_i = \underline{b}_i$ $(i = 1,2,3)$ prior to the rotation; determine C_{ij} by evaluating $\underline{a}_i \cdot \underline{b}_j$; and use Eqs. (15) - (18) to form ε_i $(i = 1,\ldots,4)$:

$$\varepsilon_4 \underset{(18)}{=} \frac{1}{2}(1 + \underline{a}_1 \cdot \underline{b}_1 + \underline{a}_2 \cdot \underline{b}_2 + \underline{a}_3 \cdot \underline{b}_3)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1 + 0 + 0 + 0)^{\frac{1}{2}} = \frac{1}{2}$$

$$\varepsilon_1 \underset{(15)}{=} \frac{\underline{a}_3 \cdot \underline{b}_2 - \underline{a}_2 \cdot \underline{b}_3}{4(1/2)} = \frac{1 - 0}{2} = \frac{1}{2}$$

$$\varepsilon_2 \underset{(16)}{=} \frac{\underline{a}_1 \cdot \underline{b}_3 - \underline{a}_3 \cdot \underline{b}_1}{4(1/2)} = \frac{-1 - 0}{2} = -\frac{1}{2}$$

$$\varepsilon_3 \underset{(17)}{=} \frac{\underline{a}_2 \cdot \underline{b}_1 - \underline{a}_1 \cdot \underline{b}_2}{4(1/2)} = \frac{-1 - 0}{2} = -\frac{1}{2}$$

Then

$$\underline{\lambda} \underset{(19)}{=} \frac{\frac{1}{2}\underline{a}_1 - \frac{1}{2}\underline{a}_2 - \frac{1}{2}\underline{a}_3}{(3/4)^{\frac{1}{2}}} = \frac{\underline{a}_1 - \underline{a}_2 - \underline{a}_3}{\sqrt{3}}$$

and

$$\theta \underset{(20)}{=} 2 \cos^{-1} (1/2) = 2\pi/3 \text{ rad.}$$

1.4 Rodrigues parameters

A vector $\underline{\rho}$, called the Rodrigues vector, and three scalar quantities, ρ_1 , ρ_2 , ρ_3 , called Rodrigues parameters, can be associated with a simple rotation of a rigid body B in a reference frame A (see Sec. 1.1) by letting

$$\underline{\rho} \triangleq \underline{\lambda} \tan(\theta/2) \tag{1}$$

and

$$\rho_i \triangleq \underline{\rho} \cdot \underline{a}_i = \underline{\rho} \cdot \underline{b}_i \qquad (i = 1,2,3) \tag{2}$$

where $\underline{\lambda}$ and θ have the same meaning as in Sec. 1.1, and $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are dextral sets of orthogonal unit vectors fixed in A and B respectively, with $\underline{a}_i = \underline{b}_i$ $(i = 1,2,3)$ prior to the rotation. (When a discussion involves more than two bodies or reference frames, notations such as ${}^A\underline{\rho}^B$ and ${}^A\rho_i^B$ will be used.)

The Rodrigues parameters are intimately related to the Euler parameters (see Sec. 1.3):

$$\rho_i = \frac{\varepsilon_i}{\varepsilon_4} \qquad (i = 1,2,3) \tag{3}$$

An advantage of the Rodrigues parameters over the Euler parameters is that they are fewer in number; but this advantage is at times offset by the fact that the Rodrigues parameters can become infinite, whereas the absolute value of any Euler parameter cannot exceed unity.

Expressed in terms of Rodrigues parameters, the direction cosine matrix C (see Sec. 1.2) assumes the form

$$C = \frac{\begin{bmatrix} 1+\rho_1^2-\rho_2^2-\rho_3^2 & 2(\rho_1\rho_2-\rho_3) & 2(\rho_3\rho_1+\rho_2) \\ 2(\rho_1\rho_2+\rho_3) & 1+\rho_2^2-\rho_3^2-\rho_1^2 & 2(\rho_2\rho_3-\rho_1) \\ 2(\rho_3\rho_1-\rho_2) & 2(\rho_2\rho_3+\rho_1) & 1+\rho_3^2-\rho_1^2-\rho_2^2 \end{bmatrix}}{1+\rho_1^2+\rho_2^2+\rho_3^2} \tag{4}$$

The Rodrigues vector can be used to establish a simple relationship between the difference and the sum of the vectors $\underline{a}$ and $\underline{b}$ defined in Sect. 1.1:

$$\underline{a} - \underline{b} = (\underline{a} + \underline{b}) \times \underline{\rho} \tag{5}$$

This relationship will be found useful in connection with a number of deviations, such as the one showing that the following is an expression for a Rodrigues vector that characterizes a simple rotation by means of which a specified change in the relative orientation of A and B can be produced:

$$\underline{\rho} = \frac{(\underline{\alpha}_1 - \underline{\beta}_1) \times (\underline{\alpha}_2 - \underline{\beta}_2)}{(\underline{\alpha}_1 + \underline{\beta}_1) \cdot (\underline{\alpha}_2 - \underline{\beta}_2)} \tag{6}$$

where $\underline{\alpha}_i$ and $\underline{\beta}_i$ $(i = 1,2)$ are vectors fixed in A and B , respectively, and $\underline{\alpha}_i = \underline{\beta}_i$ $(i = 1,2)$ prior to the change in relative orientation. Rodrigues parameters for such a rotation can be expressed as

$$\rho_1 = \frac{C_{31} + C_{13}}{C_{21} - C_{12}} \tag{7}$$

$$\rho_2 = \frac{C_{12} + C_{21}}{C_{32} - C_{23}} \tag{8}$$

$$\rho_3 = \frac{C_{23} + C_{32}}{C_{12} - C_{31}} \tag{9}$$

Derivations: The equality of $\underline{\rho} \cdot \underline{a}_i$ and $\underline{\rho} \cdot \underline{b}$ [see Eq. (2)] follows from Eqs. (1) and (1.2.22); and Eqs. (3) are obtained by noting that

$$\frac{\underline{\varepsilon}}{\varepsilon_4} \underset{(1.3.1,1.3.3)}{=} \underline{\lambda} \tan(\theta/2) \underset{(1)}{=} \underline{\rho}$$

so that

$$\rho_i \underset{(2)}{=} \frac{\underline{\varepsilon} \cdot \underline{a}_i}{\varepsilon_4} \underset{(1.3.2)}{=} \frac{\varepsilon_i}{\varepsilon_4} \qquad (i = 1,2,3)$$

From Eq. (1.3.4) ,

$$1 = \varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2 + \varepsilon_4{}^2 \underset{(3)}{=} (\rho_1{}^2 + \rho_2{}^2 + \rho_3{}^2 + 1)\varepsilon_4{}^2$$

Hence

$$\varepsilon_4{}^2 = \frac{1}{1 + \rho_1{}^2 + \rho_2{}^2 + \rho_3{}^2}$$

and

$$C_{11} \underset{(1.3.6)}{=} \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2$$

$$\underset{(3)}{=} \rho_1^2\varepsilon_4^2 - \rho_2^2\varepsilon_4^2 - \rho_3^2\varepsilon_4^2 + \varepsilon_4^2$$

$$= (\rho_1^2 - \rho_2^2 - \rho_3^2 + 1)\varepsilon_4^2 = \frac{\rho_1^2 - \rho_2^2 - \rho_3^2 + 1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2}$$

in agreement with Eq. (4) and the remaining elements of C are found similarily.

As for Eq. (5), note that cross-multiplication of Eq. (1.3.21) with $\underline{\varepsilon}$ yields

$$\underline{\varepsilon} \times \underline{b} = \underline{\varepsilon} \times \underline{a} + 2\,\{\varepsilon_4\underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a}) + \underline{\varepsilon} \times [\underline{\varepsilon} \times \underline{a})] \tag{a}$$

Hence

$$\underline{\varepsilon} \times (\underline{a} + \underline{b}) = \underline{\varepsilon} \times \underline{a} + \underline{\varepsilon} \times \underline{b}$$

$$\underset{(a)}{=} 2\{\underline{\varepsilon} \times \underline{a} + \varepsilon_4\underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a}) + \underline{\varepsilon} \times [\underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a})]\} \tag{b}$$

Now

$$\underline{\varepsilon} \times [\underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a})] = -\underline{\varepsilon}^2\underline{\varepsilon} \times \underline{a} \underset{(1.3.2)}{=} -(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)\underline{\varepsilon} \times \underline{a}$$

$$\underset{(1.3.4)}{=} (\varepsilon_4^2 - 1)\underline{\varepsilon} \times \underline{a} \tag{c}$$

Consequently

$$\underline{\varepsilon} \times (\underline{a} + \underline{b}) \underset{(b,c)}{=} 2\varepsilon_4[\varepsilon_4\underline{\varepsilon} \times \underline{a} + \underline{\varepsilon} \times (\underline{\varepsilon} \times \underline{a}]$$

$$\underset{(1.3.21)}{=} \varepsilon_4(\underline{b} - \underline{a}) \tag{d}$$

and

$$\underline{a} - \underline{b} \underset{(d)}{=} (\underline{a} + \underline{b}) \times (\underline{\varepsilon}/\varepsilon_4)$$

$$\underset{(1.3.1,1.3.2)}{=} (\underline{a} + \underline{b}) \times \underline{\lambda} \tan(\theta/2) \underset{(1)}{=} (\underline{a} + \underline{b}) \times \underline{\rho}$$

which is Eq. (5).

Eq. (6) can now be obtained by observing that

$$\underline{\alpha}_1 - \underline{\beta}_1 \underset{(5)}{=} (\underline{\alpha}_1 + \underline{\beta}_1) \times \underline{\rho} \tag{e}$$

and

$$\underline{\alpha}_2 - \underline{\beta}_2 \underset{(5)}{=} (\underline{\alpha}_2 + \underline{\beta}_2) \times \underline{\rho} \tag{f}$$

so that

$$(\underline{\alpha}_1 - \underline{\beta}_1) \times (\underline{\alpha}_2 - \underline{\beta}_2) \underset{(e)}{=} [(\underline{\alpha}_1 + \underline{\beta}_1) \times \underline{\rho}] \times (\underline{\alpha}_2 - \underline{\beta}_2)$$

$$= (\underline{\alpha}_1 + \underline{\beta}_1) \cdot (\underline{\alpha}_2 - \underline{\beta}_2)\underline{\rho} - (\underline{\alpha}_1 + \underline{\beta}_1)\underline{\rho} \cdot (\underline{\alpha}_2 - \underline{\beta}_2)$$

$$\underset{(f)}{=} (\underline{\alpha}_1 + \underline{\beta}_1) \cdot (\underline{\alpha}_2 - \underline{\beta}_2)\underline{\rho} - 0$$

from which Eq. (6) follows immediately.

Finally, to establish the validity of Eqs. (7) - (9), take

$$\underline{\alpha}_1 = \underline{a}_1 \quad , \quad \underline{\alpha}_2 = \underline{a}_2 \tag{g}$$

and hence

$$\underline{\beta}_1 = \underline{b}_1 \underset{(1.2.2)}{=} C_{11}\underline{a}_1 + C_{21}\underline{a}_2 + C_{31}\underline{a}_3 \tag{h}$$

and

$$\underline{\beta}_2 = \underline{b}_2 \underset{(1.2.2)}{=} C_{12}\underline{a}_1 + C_{22}\underline{a}_2 + C_{32}\underline{a}_3 \tag{i}$$

Then

$$\begin{aligned}(\underline{\alpha}_1 + \underline{\beta}_1) \cdot (\underline{\alpha}_2 - \underline{\beta}_2) &\underset{(g-i)}{=} C_{21} - C_{12} - (C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32}) \\ &\underset{(1.2.14)}{=} C_{21} - C_{12}\end{aligned} \tag{j}$$

and

$$\begin{aligned}(\underline{\alpha}_1 - \underline{\beta}_1) \times (\underline{\alpha}_2 - \underline{\beta}_2) \underset{(g-i)}{=} & [C_{21}C_{32} + C_{31}(1 - C_{22})]\underline{a}_1 \\ & + [C_{31}C_{12} + C_{32}(1 - C_{11})]\underline{a}_2 \\ & + [(1 - C_{11})(1 - C_{22}) - C_{21}C_{12}]\underline{a}_3\end{aligned} \tag{k}$$

so that

$$\begin{aligned}\rho_1 &\underset{(2)}{=} \underline{\rho} \cdot \underline{a}_1 \underset{(5)}{=} \frac{(\underline{\alpha}_1 - \underline{\beta}_1) \times (\underline{\alpha}_2 - \underline{\beta}_2)}{(\underline{\alpha}_1 + \underline{\beta}_1) \cdot (\underline{\alpha}_1 - \underline{\beta}_2)} \cdot \underline{a}_1 \\ &\underset{(j,k)}{=} \frac{C_{21}C_{32} - C_{31}C_{22} + C_{31}}{C_{21} - C_{12}}\end{aligned} \tag{ℓ}$$

Now, $C_{21}C_{32} - C_{31}C_{22}$ is the cofactor of C_{13} in the determinant of C, and is thus equal to C_{13} (see Sec. 1.2). Consequently, Eq. (ℓ) is equivalent to Eq. (7); and Eqs. (8) and (9) can be obtained by cyclic permutation of the subscripts in Eq. (7).

Example: Referring to Fig. 1.4.1, which depicts the rigid body B previously considered in the Example in Sec. 1.1, suppose that B is subjected to a one-hundred-and-eighty degree rotation relative to A about an axis parallel to the unit vector $\underline{\lambda}$; and let $\underline{b}_i$ $(i = 1,2,3)$ be a unit vector fixed in B and equal to $\underline{a}_i$ $(i = 1,2,3)$ prior to the rotation. The direction cosine matrix C satisfying Eq. (1.2.2) subsequent to the rotation is to be determined.

For $\theta = \pi$ rad., Eq. (1) yields a Rodrigues vector of infinite magnitude, and Eqs. (2) and (4) lead to an indeterminate form of C . To evaluate this indeterminate form, one may express each element of C in terms of θ and $\underline{\lambda} \cdot \underline{a}_i$ $(i = 1,2,3)$ by reference to Eqs. (1) and (2), and then determine the limit approached by the resulting expression as θ approaches π rad. Alternativly, one can use either Eqs. (1.2.23) - (1.2.31) or Eqs. (1.3.6) - (1.3.14) to find the elements of C . The latter equations are particularly convenient, because, for $\theta = \pi$,

$$\underline{\varepsilon} \underset{(1.3.1)}{=} \underline{\lambda}$$

so that

$$\varepsilon_i \underset{(1.3.2)}{=} \underline{\lambda} \cdot \underline{a}_i \qquad (i = 1,2,3)$$

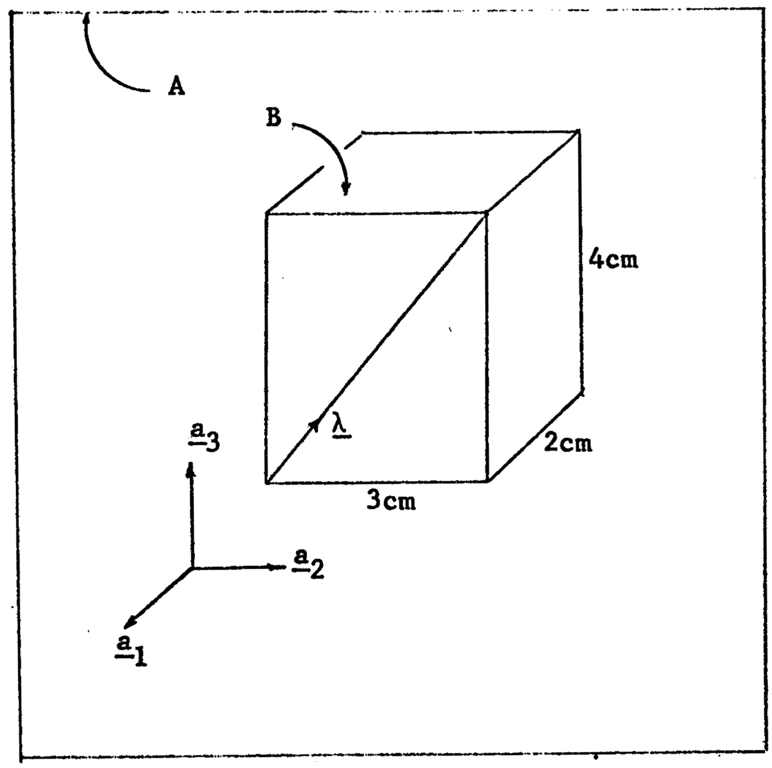
A
B
4cm
$\underline{\lambda}$
$\underline{a}_3$
2cm
3cm
$\underline{a}_2$
$\underline{a}_1$

Figure 1.4.1

and

$$\varepsilon_4 \underset{(1.3.3)}{=} 0$$

Hence, with

$$\underline{\lambda} = \frac{3}{5}\,\underline{a}_2 + \frac{4}{5}\,\underline{a}_3$$

one finds immediately that

$$\varepsilon_1 = 0\ , \qquad \varepsilon_2 = \frac{3}{5}\ , \qquad \varepsilon_3 = \frac{4}{5}$$

and substitution into Eqs. (1.3.6) - (11.3.14) then gives

$$C = \frac{1}{25}\begin{bmatrix} -25 & 0 & 0 \\ 0 & -7 & 24 \\ 0 & 24 & 7 \end{bmatrix}$$

1.5 Indirect determination of orientation

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are dextral sets of orthogonal unit vectors fixed in reference frames A and B respectively, and the orientation of each unit vector in both reference frames is known, then a description of the relative orientation of the two reference frames can be given in terms of direction cosines C_{ij} $(i,j = 1,2,3)$, for, in accordance with Eq. (1.2.1), these can be found by simply evaluating $\underline{a}_i \cdot \underline{b}_j$ $(i,j = 1,2,3)$. But, even when these dot-products cannot be evaluated so directly, it may be possible to find C_{ij} $(i,j = 1,2,3)$. This is the case, for example, when each of two non-parallel vectors, say $\underline{p}$ and $\underline{q}$, has a known orientation in both A and B , so that the dot-products $\underline{a}_i \cdot \underline{p}$, $\underline{a}_i \cdot \underline{q}$, $\underline{b}_i \cdot \underline{p}$, and $\underline{b}_i \cdot \underline{q}$ $(i = 1,2,3)$ can be evaluated directly. In that event, one can find C_{ij} as follows: Form a vector $\underline{r}$ and a dyadic $\underline{\sigma}$ by letting

$$\underline{r} \triangleq \underline{p} \times \underline{q} \tag{1}$$

and

$$\underline{\sigma} \triangleq (\underline{p} \times \underline{q}\underline{r} + \underline{q} \times \underline{r}\underline{p} + \underline{r} \times \underline{p}\underline{q})/\underline{r}^2 \tag{2}$$

Next, express the first member of each dyad in Eq. (2) in terms of $\underline{a}_i$, and the second member in terms of $\underline{b}_i$ $(i = 1,2,3)$. Finally, carry out the multiplications indicated in the relationship

$$C_{ij} = \underline{a}_i \cdot \underline{\sigma} \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{3}$$

<u>Deviation</u>: It will be shown that the dyadic $\underline{\sigma}$ defined in Eq. (2) is a unit dyadic, that is, that for every vector $\underline{v}$,

$$\underline{v} = \underline{v} \cdot \underline{\sigma}$$

The validity of Eq. (3) can then be seen to be an immediate consequence of the definition of C_{ij} , given in Eq. (1.1.1).

If $\underline{p}$ and $\underline{q}$ are non-parallel, and $\underline{r}$ is defined as in Eq. (1), then every vector $\underline{v}$ can be expressed as

$$\underline{v} = \alpha\underline{p} + \beta\underline{q} + \gamma\underline{r} \tag{a}$$

where α , β , and γ are certain scalars. From Eq. (a),

$$\begin{aligned} \underline{v} \cdot (\underline{q} \times \underline{r}) &= \alpha\underline{p} \cdot (\underline{q} \times \underline{r}) + \beta\underline{q} \cdot (\underline{q} \times \underline{r}) + \gamma\underline{r} \cdot (\underline{q} \times \underline{r}) \\ &= \alpha(\underline{p} \times \underline{q}) \cdot \underline{r} + 0 + 0 \\ &\underset{(1)}{=} \alpha\underline{r}^2 \end{aligned} \tag{b}$$

so that

$$\alpha \underset{(b)}{=} \underline{v} \cdot (\underline{q} \times \underline{r})/\underline{r}^2 \tag{c}$$

Similarily, scalar multiplication of Eq. (a) with $\underline{r} \times \underline{p}$ and $\underline{p} \times \underline{q}$ leads to the conclusion that

$$\beta = \underline{v} \cdot (\underline{r} \times \underline{p})/\underline{r}^2 \tag{d}$$

and

$$\gamma = \underline{v} \cdot (\underline{p} \times \underline{q})/\underline{r}^2 \tag{e}$$

Substituting from Eqs. (c) - (d) into Eq. (a), one thus finds that

$$\underline{v} = [\underline{v} \cdot (\underline{q} \times \underline{r})\underline{p} + \underline{v} \cdot (\underline{r} \times \underline{p})\underline{q} + \underline{v} \cdot (\underline{p} \times \underline{q})\underline{r}]/\underline{r}^2$$

$$= \underline{v} \cdot [(\underline{q} \times \underline{rp} + \underline{r} \times \underline{pq} + \underline{p} \times \underline{qr})/\underline{r}^2] \underset{(2)}{=} \underline{v} \cdot \underline{\sigma}$$

Example: Observations of two stars, P and Q, are made simultaneously from two space vehicles, A and B, in order to generate data to be used in the determination of the relative orientation of A and B. The observations consist of measuring the angles ϕ and ψ shown in Fig. 1.5.1, where O represents either a point fixed in A or a point fixed in B, R is either P or Q, and $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ are orthogonal unit vectors forming a dextral set fixed either in A or B. For the numerical values of these angles given in Table 1, the direction cosine matrix C is to be determined.

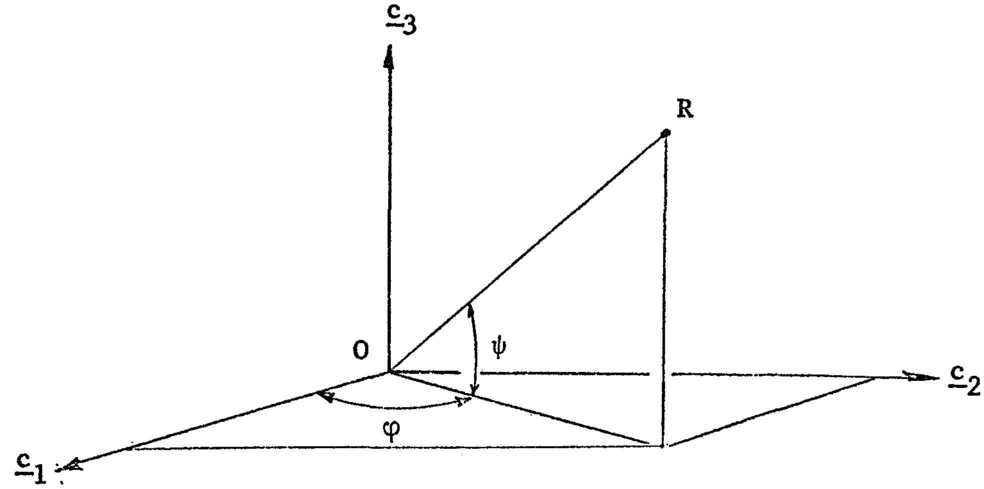

Figure 1.5.1

Table 1

Angles ϕ and ψ in degrees

	P		Q	
	ϕ	ψ	ϕ	ψ
$\underline{c}_i = \underline{a}_i$	90	45	30	0
$\underline{c}_i = \underline{b}_i$	135	0	90	60

If $\underline{R}$ is defined as a unit vector directed from 0 toward R (see Fig. 1.5.1), then

$$\underline{R} = \cos\psi \cos\phi \underline{c}_1 + \cos\psi \sin\phi \underline{c}_2 + \sin\psi \underline{c}_3$$

Hence, letting $\underline{p}$ and $\underline{q}$ be unit vectors directed from 0 toward P and toward Q, respectively, and referring to Table 1, one can express each of these unit vectors both in terms of $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and in terms of $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, as indicated in lines 1 and 2 of Table 2; and these results can then be used to evaluate $\underline{r}$ [see Eq. (1)], $\underline{q} \times \underline{r}$, and $\underline{r} \times \underline{p}$. Noting that (see line 3 of Table 2)

Table 2

Vectors appearing in $\underline{\sigma}$

line	vector	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}_3$
1	$\underline{p}$	0	$\sqrt{2}/2$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	0
2	$\underline{q}$	$\sqrt{3}/2$	1/2	0	0	1/2	$\sqrt{3}/2$
3	$\underline{r} = \underline{p} \times \underline{q}$	$-\sqrt{2}/4$	$\sqrt{6}/4$	$-\sqrt{6}/4$	$\sqrt{6}/4$	$\sqrt{6}/4$	$-\sqrt{2}/4$
4	$\underline{q} \times \underline{r}$	$-\sqrt{6}/8$	$3\sqrt{2}/8$	$\sqrt{2}/2$	-	-	-
5	$\underline{r} \times \underline{p}$	$\sqrt{3}/2$	1/4	$-1/4$	-	-	-

$$\underline{r}^2 = \frac{2}{16} + \frac{6}{16} + \frac{6}{16} = \frac{7}{8}$$

one thus obtains

$$\underset{=}{\underline{\sigma}} \underset{(2)}{=} \Bigg[\left(-\frac{\sqrt{2}}{4}\underline{a}_1 + \frac{\sqrt{6}}{4}\underline{a}_2 - \frac{\sqrt{6}}{4}\underline{a}_3\right)\left(\frac{\sqrt{6}}{4}\underline{b}_1 + \frac{\sqrt{6}}{4}\underline{b}_2 - \frac{\sqrt{2}}{4}\underline{b}_3\right)$$

$$+ \left(-\frac{\sqrt{6}}{8}\underline{a}_1 + \frac{3\sqrt{2}}{8}\underline{a}_2 + \frac{\sqrt{2}}{2}\underline{a}_3\right)\left(-\frac{\sqrt{2}}{2}\underline{b}_1 - \frac{\sqrt{2}}{2}\underline{b}_2 + 0\underline{b}_3\right)$$

$$+ \left(\frac{\sqrt{3}}{2}\underline{a}_1 + \frac{1}{4}\underline{a}_2 - \frac{1}{4}\underline{a}_3\right)\left(0\underline{b}_1 + \frac{1}{2}\underline{b}_2 + \frac{\sqrt{3}}{2}\underline{b}_3\right)\Bigg] \Big/ \left(\frac{7}{8}\right)$$

and

$$C_{11} \underset{(3)}{=} \underline{a}_1 \cdot \underset{=}{\underline{\sigma}} \cdot \underline{b}_1 = \left(-\frac{\sqrt{2}}{4}\frac{\sqrt{6}}{4} + \frac{\sqrt{6}}{8}\frac{\sqrt{2}}{2}\right) \Big/ \left(\frac{7}{8}\right) = 0$$

$$C_{12} \underset{(3)}{=} \underline{a}_1 \cdot \underset{=}{\underline{\sigma}} \cdot \underline{b}_2 = \left(-\frac{\sqrt{2}}{4}\frac{\sqrt{6}}{4} - \frac{\sqrt{6}}{8}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{1}{2}\right) \Big/ \left(\frac{7}{8}\right) = 0$$

$$C_{13} \underset{(3)}{=} \underline{a}_1 \cdot \underset{=}{\underline{\sigma}} \cdot \underline{b}_3 = \left(\frac{\sqrt{2}}{4}\frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\right) \Big/ \left(\frac{7}{8}\right) = 1$$

and so forth; that is,

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

1.6 Successive rotations

When a rigid body B is subjected to two successive simple rotations (see Sec. 1.1) in a reference frame A , each of these rotations, as well as a single equivalent rotation (see Sec. 1.3), can be described in terms of direction cosines (see Sec. 1.2), Euler parameters (see Sec. 1.3) and Rodrigues parameters (see Sec. 1.4); and, no matter which method of description is employed, quantities associated with the individual rotations can be related to those characterizing the single equivalent rotation. In discussing such relationships, it is helpful to introduce a fictitious rigid body $\bar{B}$ which moves exactly like B during the first rotation, but remains fixed in A while B performs the second rotation. For analytical purposes, the first rotation can then be regarded as a rotation of $\bar{B}$ relative to A , and the second rotation as one of B relative to $\bar{B}$.

If $\underline{a}_i$ $(i = 1,2,3)$, $\underline{b}_i$ $(i = 1,2,3)$, and $\underline{\bar{b}}_i$ $(i = 1,2,3)$ are three dextral sets of orthogonal unit vectors fixed in A , B , and $\bar{B}$ respectively, and such that $\underline{a}_i = \underline{b}_i = \underline{\bar{b}}_i$ $(i = 1,2,3)$ prior to the first rotation of B in A , and if ${}^{A}C^{\bar{B}}$, ${}^{\bar{B}}C^{B}$, and ${}^{A}C^{B}$ are the associated direction cosign matrices characterizing respectively the first, the second, and the single equivalent rotation, so that

$$[\underline{\bar{b}}_1\ \underline{\bar{b}}_2\ \underline{\bar{b}}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]\ {}^{A}C^{\bar{B}} \tag{1}$$

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{\bar{b}}_1\ \underline{\bar{b}}_2\ \underline{\bar{b}}_3]\ {}^{\bar{B}}C^{B} \tag{2}$$

and

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]\ {}^{A}C^{B} \tag{3}$$

then ${}^{A}C^{B}$, expressed in terms of ${}^{A}C^{\overline{B}}$ and ${}^{\overline{B}}C^{B}$, is given by

$$ {}^{A}C^{B} = {}^{A}C^{\overline{B}}\ {}^{\overline{B}}C^{B} \tag{4}$$

Similarly, for Rodrigues vectors,

$$ {}^{A}\underline{\rho}^{B} = \frac{{}^{A}\underline{\rho}^{\overline{B}} + {}^{\overline{B}}\underline{\rho}^{B} + {}^{\overline{B}}\underline{\rho}^{B} \quad {}^{A}\underline{\rho}^{\overline{B}}}{1 - {}^{A}\underline{\rho}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\rho}^{B}} \tag{5}$$

To state the analogous relationship in terms of Euler parameters, we first define three sets of such parameters as follows: With $\underline{\overline{b}}_i$ and $\underline{b}_i$ $(i = 1,2,3)$ directed as after the second rotation, and with θ_1 , θ_2 , and θ denoting respectively the radian measures of the first, the second, and the equivalent rotation,

$$ {}^{A}\varepsilon_i^{\overline{B}} \triangleq {}^{A}\underline{\varepsilon}^{\overline{B}} \cdot \underline{a}_i = {}^{A}\underline{\varepsilon}^{\overline{B}} \cdot \underline{\overline{b}}_i \quad (i = 1,2,3) \tag{6}$$

$$ {}^{\overline{B}}\varepsilon_i^{B} \triangleq {}^{\overline{B}}\underline{\varepsilon}^{B} \cdot \underline{\overline{b}}_i = {}^{\overline{B}}\underline{\varepsilon}^{B} \cdot \underline{b}_i \quad (i = 1,2,3) \tag{7}$$

$$ {}^{A}\varepsilon_i^{B} \triangleq {}^{A}\underline{\varepsilon}^{B} \cdot \underline{a}_i = {}^{A}\underline{\varepsilon}^{B} \cdot \underline{b}_i \quad (i = 1,2,3) \tag{8}$$

$$ {}^{A}\varepsilon_4^{\overline{B}} \triangleq \cos\ (\theta_1/2) \tag{9}$$

$$ {}^{\overline{B}}\varepsilon_4^{B} \triangleq \cos\ (\theta_2/2) \tag{10}$$

$$ {}^{A}\varepsilon_4^{B} \triangleq \cos\ (\theta/2) \tag{11}$$

It then follows that

$$\begin{bmatrix} {}^{A}\epsilon_1{}^{B} \\ {}^{A}\epsilon_2{}^{B} \\ {}^{A}\epsilon_3{}^{B} \\ {}^{A}\epsilon_4{}^{B} \end{bmatrix} = \begin{bmatrix} {}^{A}\epsilon_4{}^{\overline{B}} & -{}^{A}\epsilon_3{}^{\overline{B}} & {}^{A}\epsilon_2{}^{\overline{B}} & {}^{A}\epsilon_1{}^{\overline{B}} \\ {}^{A}\epsilon_3{}^{\overline{B}} & {}^{A}\epsilon_4{}^{\overline{B}} & -{}^{A}\epsilon_1{}^{\overline{B}} & {}^{A}\epsilon_2{}^{\overline{B}} \\ -{}^{A}\epsilon_2{}^{\overline{B}} & {}^{A}\epsilon_1{}^{\overline{B}} & {}^{A}\epsilon_4{}^{\overline{B}} & {}^{A}\epsilon_3{}^{\overline{B}} \\ -{}^{A}\epsilon_1{}^{\overline{B}} & -{}^{A}\epsilon_2{}^{\overline{B}} & -{}^{A}\epsilon_3{}^{\overline{B}} & {}^{A}\epsilon_4{}^{\overline{B}} \end{bmatrix} \begin{bmatrix} {}^{\overline{B}}\epsilon_1{}^{B} \\ {}^{\overline{B}}\epsilon_2{}^{B} \\ {}^{\overline{B}}\epsilon_3{}^{B} \\ {}^{\overline{B}}\epsilon_4{}^{B} \end{bmatrix} \tag{12}$$

Furthermore,

$${}^{A}\underline{\epsilon}^{B} = {}^{A}\underline{\epsilon}^{\overline{B}}\,{}^{\overline{B}}\epsilon_4{}^{B} + {}^{\overline{B}}\underline{\epsilon}^{B}\,{}^{A}\epsilon_4{}^{\overline{B}} + {}^{\overline{B}}\underline{\epsilon}^{B} \times {}^{A}\underline{\epsilon}^{\overline{B}} \tag{13}$$

and

$${}^{A}\epsilon_4{}^{B} = {}^{A}\epsilon_4{}^{\overline{B}}\,{}^{\overline{B}}\epsilon_4{}^{B} - {}^{A}\underline{\epsilon}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\epsilon}^{\overline{B}} \tag{14}$$

Eqs. (4), (5), (12), and (13) all reflect the fact that the final orientation of B in A depends upon the order in which the successive rotations are performed. For example, in Eq. (4), ${}^{A}C^{\overline{B}}$ and ${}^{\overline{B}}C^{B}$ cannot be interchanged without altering the result, and in Eq. (13) the presence of a cross-product shows that order cannot be left out of account.

Repeated use of Eqs. (4) - (14) permits one to construct formulas for quantities characterizing a single rotation that is equivalent to any number of successive rotations. For example, for three successive rotations,

$$ {}^{A}C^{B} = {}^{A}C^{\overline{B}}\;{}^{\overline{B}}C^{\overline{\overline{B}}}\;{}^{\overline{\overline{B}}}C^{B} \tag{15}$$

where ${}^{A}C^{\overline{B}}$, ${}^{\overline{B}}C^{\overline{\overline{B}}}$, and ${}^{\overline{\overline{B}}}C^{B}$ are direction cosine matrices associated with the first, the second, and the third rotation, respectively.

<u>Derivations</u>: Substitution from Eq. (1) into Eq. (2) gives

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]\,{}^{A}C^{\overline{B}}\,{}^{\overline{B}}C^{B}$$

and comparison of this equation with Eq. (3) shows that Eq. (4) is valid.

To obtain Eq. (5), let $\underline{a}$, $\underline{\overline{b}}$, and $\underline{b}$ be vectors fixed in A , $\overline{B}$, and B , respectively, and choose these in such a way that $\underline{a} = \underline{\overline{b}} = \underline{b}$ prior to the first rotation of B in A . Then, in accordance with Eq. (1.4.5), there exist Rodrigues vectors ${}^{A}\underline{\rho}^{\overline{B}}$, ${}^{\overline{B}}\underline{\rho}^{B}$, and ${}^{A}\underline{\rho}^{B}$ satisfying the equations

$$\underline{a} - \underline{\overline{b}} = (\underline{a} + \underline{\overline{b}}) \times {}^{A}\underline{\rho}^{\overline{B}} \tag{a}$$

$$\underline{\overline{b}} - \underline{b} = (\underline{\overline{b}} + \underline{b}) \times {}^{\overline{B}}\underline{\rho}^{B} \tag{b}$$

and

$$\underline{a} - \underline{b} = (\underline{a} + \underline{b}) \times {}^{A}\underline{\rho}^{B} \tag{c}$$

Cross-multiply Eqs. (a) and (b) with ${}^{\overline{B}}\underline{\rho}^{B}$ and ${}^{A}\underline{\rho}^{\overline{B}}$, respectively;

subtract the resulting equations; and use the fact that

$$ {}^{A}\underline{\rho}^{\overline{B}} \cdot \underline{a} = {}^{A}\underline{\rho}^{\overline{B}} \cdot \overline{\underline{b}} $$

and

$$ {}^{\overline{B}}\underline{\rho}^{B} \cdot \overline{\underline{b}} = {}^{\overline{B}}\underline{\rho}^{B} \cdot \underline{b} $$

to eliminate $\overline{\underline{b}}$ wherever it appears as a member of a scalar product. This leads to

$$ \overline{\underline{b}} \times \left({}^{A}\underline{\rho}^{\overline{B}} + {}^{\overline{B}}\underline{\rho}^{B}\right) = \underline{a} \times {}^{\overline{B}}\underline{\rho}^{B} + \underline{b} \times {}^{A}\underline{\rho}^{\overline{B}} + (\underline{a} - \underline{b})\,{}^{A}\underline{\rho}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\rho}^{B} $$

$$ + (\underline{a} + \underline{b}) \times \left({}^{\overline{B}}\underline{\rho}^{B} \times {}^{A}\underline{\rho}^{\overline{B}}\right) \tag{d} $$

Next, add Eqs. (a) and (b), and eliminate $\overline{\underline{b}}$ by using Eq. (d), thus obtaining

$$ \underline{a} - \underline{b} = (\underline{a} + \underline{b}) \times \frac{{}^{A}\underline{\rho}^{\overline{B}} + {}^{\overline{B}}\underline{\rho}^{B} + {}^{\overline{B}}\underline{\rho}^{B} \quad {}^{A}\underline{\rho}^{\overline{B}}}{1 - {}^{A}\underline{\rho}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\rho}^{B}} \tag{e} $$

Together, Eqs. (e) and (c) imply the validity of Eq. (5), for Eqs. (e) and (c) can be satisfied for all choices of the vector $\underline{a}$ only if Eq. (5) is satisfied.

As for Eqs. (13), and (14), note that it follows from Eqs. (1.3.1), (1.3.3), and (1.4.1) that

$$ {}^{A}\underline{\rho}^{\overline{B}} = {}^{A}\underline{\varepsilon}^{\overline{B}} \Big/ {}^{A}\varepsilon_{4}^{\overline{B}} \tag{f} $$

$$ {}^{\bar{B}}\underline{\rho}^{B} = {}^{\bar{B}}\underline{\varepsilon}^{B} \Big/ {}^{\bar{B}}\varepsilon_4^{B} \tag{g} $$

and

$$ {}^{A}\underline{\rho}^{B} = {}^{A}\underline{\varepsilon}^{B} \Big/ {}^{A}\varepsilon_4^{B} \tag{h} $$

Consequently,

$$ \begin{aligned} {}^{A}\underline{\varepsilon}^{B} &\underset{(h)}{=} {}^{A}\varepsilon_4^{B}\ {}^{A}\underline{\rho}^{B} \underset{(5)}{=} {}^{A}\varepsilon_4^{B}\ \frac{{}^{A}\underline{\rho}^{\bar{B}} + {}^{\bar{B}}\underline{\rho}^{B} + {}^{\bar{B}}\underline{\rho}^{B} \times {}^{A}\underline{\rho}^{\bar{B}}}{1 - {}^{A}\underline{\rho}^{\bar{B}} \cdot {}^{\bar{B}}\underline{\rho}^{B}} \\ &\underset{(f,g)}{=} {}^{A}\varepsilon_4^{B}\ \frac{{}^{A}\underline{\varepsilon}^{\bar{B}}\ {}^{\bar{B}}\varepsilon_4^{B} + {}^{\bar{B}}\underline{\varepsilon}^{B}\ {}^{A}\varepsilon_4^{\bar{B}} + {}^{\bar{B}}\underline{\varepsilon}^{B} \times {}^{A}\underline{\varepsilon}^{\bar{B}}}{{}^{A}\varepsilon_4^{\bar{B}}\ {}^{\bar{B}}\varepsilon_4^{B} - {}^{A}\underline{\varepsilon}^{\bar{B}} \cdot {}^{\bar{B}}\underline{\varepsilon}^{B}} \end{aligned} \tag{i} $$

and, dot-multiplying each side of this equation with itself and using Eq. (1.3.4), one finds that

$$ \left({}^{A}\underline{\varepsilon}^{B}\right)^2 = \left({}^{A}\varepsilon_4^{B}\right)^2 \frac{1 - \left({}^{A}\varepsilon_4^{\bar{B}}\ {}^{\bar{B}}\varepsilon_4^{B} - {}^{A}\underline{\varepsilon}^{\bar{B}} \cdot {}^{\bar{B}}\underline{\varepsilon}^{B}\right)^2}{\left({}^{A}\varepsilon_4^{\bar{B}}\ {}^{\bar{B}}\varepsilon_4^{B} - {}^{A}\underline{\varepsilon}^{\bar{B}} \cdot {}^{\bar{B}}\underline{\varepsilon}^{B}\right)^2} \tag{j} $$

But

$$ \left({}^{A}\underline{\varepsilon}^{B}\right)^2 \underset{(1.3.4)}{=} 1 - \left({}^{A}\varepsilon_4^{B}\right)^2 \tag{k} $$

Hence

$$\left({}^{A}\varepsilon_4{}^{B}\right)^2 \underset{(j,k)}{=} \left({}^{A}\varepsilon_4{}^{B}\,{}^{\overline{B}}\varepsilon_4{}^{B} - {}^{A}\underline{\varepsilon}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\varepsilon}^{B}\right)^2 \qquad (\ell)$$

and

$$^{A}\varepsilon_4{}^{B} \underset{(\ell)}{=} \pm\left({}^{A}\varepsilon_4{}^{\overline{B}}\,{}^{B}\varepsilon_4{}^{B} - {}^{A}\underline{\varepsilon}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\varepsilon}^{B}\right) \qquad (m)$$

so that, using the upper sign, one obtains Eq. (14). Furthermore, substitution into Eq. (i) then yields Eq. (13).

Finally, to establish the validity of Eq. (12), it sufficies to show that the four scalar equations implied by this matrix equation can be derived from Eqs. (13) and (14). To this end, one may employ Eqs. (6) and (7) to resolve the right-hand member of Eq. (13) into components parallel to $\overline{\underline{b}}_1$, $\overline{\underline{b}}_2$, and $\overline{\underline{b}}_3$ and then dot-multiply both sides of the resulting equation successively with $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$, using Eq. (9) to evaluate ${}^{A}\underline{\varepsilon}^{B} \cdot \underline{b}_i$ and Eqs. (1.3.6) - (1.3.14) to form $\underline{b}_i \cdot \underline{b}_j$, which gives, for example,

$$\overline{\underline{b}}_2 \cdot \underline{b}_3 \underset{(1.3.11)}{=} 2\left({}^{\overline{B}}\varepsilon_2{}^{B}\,{}^{\overline{B}}\varepsilon_3{}^{B} - {}^{\overline{B}}\varepsilon_1{}^{B}\,{}^{\overline{B}}\varepsilon_4{}^{B}\right)$$

In this way one is led to the first three scalar equations corresponding to Eq. (12); and the fourth is obtained from Eq. (14) by making the substitution

$$^{A}\underline{\varepsilon}^{\overline{B}} \cdot {}^{\overline{B}}\underline{\varepsilon}^{B} \underset{(6,7)}{=} {}^{A}\varepsilon_1{}^{\overline{B}}\,{}^{\overline{B}}\varepsilon_1{}^{B} + {}^{A}\varepsilon_2{}^{\overline{B}}\,{}^{\overline{B}}\varepsilon_2{}^{B} + {}^{A}\varepsilon_3{}^{\overline{B}}\,{}^{\overline{B}}\varepsilon_3{}^{B}$$

<u>Example</u>: In Fig. 1.6.1, $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are mutually perpendicular unit vectors; X and Y are lines perpendicular to $\underline{a}_1$, and making fixed angles with $\underline{a}_2$ and $\underline{a}_3$; and B designates a body that is to be subjected to a ninety degree rotation about line X and a one-hundred-and-eighty degree rotation about line Y , the sense of each of these rotations being that indicated in the sketch.

Suppose that the rotation about X is performed first. Then, if $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are unit vectors fixed in B , there exists a matrix C_x such that, subsequent to the second rotation of B ,

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]C_x$$

Similarly, if the rotation about Y is performed first, there exists a matrix C_y such that, subsequent to the second rotation of B ,

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]C_y$$

C_x and C_y are to be determined.

In order to find C_x by using Eq. (4), one may first form ${}^{A}C^{\overline{B}}$ by reference to Eqs. (1.2.23) - (1.2.31) with $\theta = \pi/2$ and

$$\lambda_1 = 0 , \qquad \lambda_2 = \sqrt{3}/2 , \qquad \lambda_3 = 1/2$$

which gives

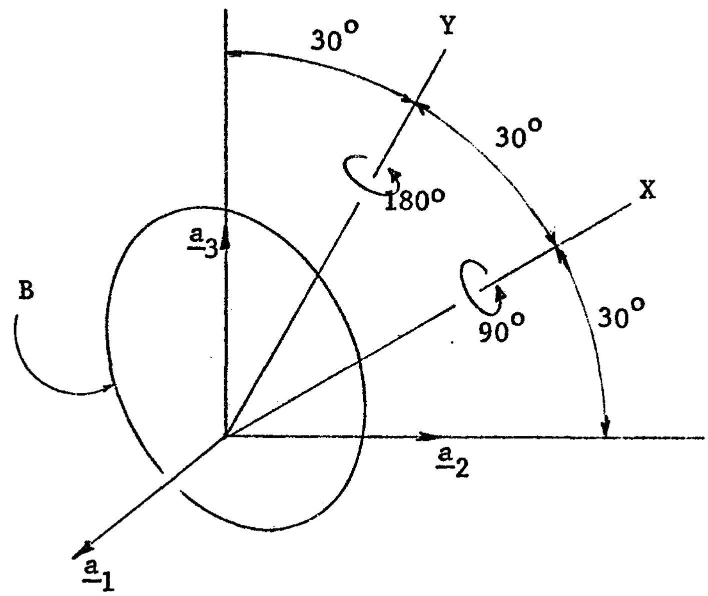

Figure 1.6.1

$$
{}^{A}C^{\bar{B}} = \begin{bmatrix} 0 & -1/2 & \sqrt{3}/2 \\ 1/2 & 3/4 & \sqrt{3}/4 \\ -\sqrt{3}/2 & \sqrt{3}/4 & 1/4 \end{bmatrix}
$$

Next, to construct the matrix ${}^{\bar{B}}C^{B}$, one must express a unit vector $\underline{\lambda}$ which is parallel to Y in terms of suitable unit vectors $\underline{\bar{b}}_1$, $\underline{\bar{b}}_2$ and $\underline{\bar{b}}_3$. This is accomplished by noting that $\underline{\lambda}$, resolved into components parallel to $\underline{a}_1$, a_2 , and $\underline{a}_3$, is given by

$$
\underline{\lambda} = \frac{1}{2}\,\underline{a}_2 + \frac{\sqrt{3}}{2}\,\underline{a}_3
$$

so that, using Eq. (1.2.9) and ${}^{A}C^{\bar{B}}$, one obtains

$$
\begin{aligned}
\underline{\lambda} &= \left[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right)\right]\underline{\bar{b}}_1 + \left[\frac{1}{2}\left(\frac{3}{4}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{4}\right)\right]\underline{\bar{b}}_2 \\
&\quad + \left[\frac{1}{2}\left(\frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{4}\right)\right]\underline{\bar{b}}_3 \\
&= -\frac{1}{2}\,\underline{\bar{b}}_1 + \frac{3}{4}\,\underline{\bar{b}}_2 + \frac{\sqrt{3}}{4}\,\underline{\bar{b}}_3
\end{aligned}
$$

With $\theta = \pi$, Eqs. (1.2.23) - (1.2.31) then provide

$$
{}^{\bar{B}}C^{B} = \begin{bmatrix} -1/2 & -3/4 & -\sqrt{3}/4 \\ -3/4 & 1/8 & 3\sqrt{3}/8 \\ -\sqrt{3}/4 & 3\sqrt{3}/8 & -5/8 \end{bmatrix}
$$

Consequently,

$$C_x \underset{(4)}{=} {}^A C^B \; {}^{\overline{B}} C^B = \begin{bmatrix} 0 & 1/2 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

C_Y can be found similarly. Alternatively, one may use Euler parameters, proceeding as follows:

With $\underline{\lambda}$ expressed in terms of $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$, and with $\theta = \pi$, Eq. (1.3.1) gives

$${}^A \underline{\varepsilon}^{\overline{B}} = \frac{1}{2}\,\underline{a}_2 + \frac{\sqrt{3}}{2}\,\underline{a}_3$$

and, from Eq. (1.3.3),

$${}^A \varepsilon_4^{\overline{B}} = 0$$

Similarly, for the second rotation

$${}^{\overline{B}} \underline{\varepsilon}^B \underset{(1.3.1)}{=} \left(\frac{\sqrt{3}}{2}\,\underline{a}_2 + \frac{1}{2}\,\underline{a}_3 \right) \frac{\sqrt{2}}{2}$$

and

$${}^{\overline{B}} \varepsilon_4^B \underset{(1.3.3)}{=} \frac{\sqrt{2}}{2}$$

Hence

$$\underline{A}_{\underline{\varepsilon}}B \underset{(13)}{=} \left(\frac{1}{2}\,\underline{a}_2 + \frac{\sqrt{3}}{2}\,\underline{a}_3\right)\frac{\sqrt{2}}{2} + 0 + \frac{\sqrt{2}}{4}\,\underline{a}_1$$

so that, in accordance with Eq. (8),

$$A_{\varepsilon_1}B = \frac{\sqrt{2}}{4}\,, \qquad A_{\varepsilon_2}B = \frac{\sqrt{2}}{4}\,, \qquad A_{\varepsilon_3}B = \frac{\sqrt{6}}{4}$$

while

$$A_{\varepsilon_4}B \underset{(11)}{=} -\frac{\sqrt{6}}{4}$$

The elements of C_y can now be obtained by using Eqs. (1.3.6) - (1.3.14), which gives

$$C_y = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

1.7 Orientation angles

Both for physical and for analytical reasons it is sometimes desirable to describe the orientation of a rigid body B in a reference frame A in terms of three angles. For example, if B is the rotor of a gyroscope whose outer gimbal axis is fixed in a reference frame A, then the angles ϕ, θ, and ψ shown in Fig. 1.7.1 furnish a means for describing the orientation of B in A in a way that is particularly meaningful from a physical point of view.

One scheme for bringing a rigid body B into a desired orientation in a reference frame A is to introduce $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ as dextral sets of orthogonal unit vectors fixed in A and B, respectively; align $\underline{b}_i$ with $\underline{a}_i$ $(i = 1,2,3)$; and subject B successively to an $\underline{a}_1$ rotation of amount θ_1, and $\underline{a}_2$ rotation of amount θ_2, and an $\underline{a}_3$ rotation of amount θ_3. (Recall that, for any unit vector $\underline{\lambda}$, the phrase "$\underline{\lambda}$ rotation" means a rotation of B relative to A during which a right-handed screw fixed in B with its axis parallel to $\underline{\lambda}$ advances in the direction of $\underline{\lambda}$.) Suitable values of θ_1, θ_2, and θ_3 can be found in terms of elements of the direction cosine matrix C (see Sec. 1.2), which, if s_i and c_i $(i = 1,2,3)$ denote $\sin\theta_i$ and $\cos\theta_i$ $(i = 1,2,3)$ respectively, is given by

$$C = \begin{bmatrix} c_2c_3 & s_1s_2c_3 - s_3c_1 & c_1s_2c_3 + s_3s_1 \\ c_2s_3 & s_1s_2s_3 + c_3c_1 & c_1s_2s_3 - c_3s_1 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix} \qquad (1)$$

Specifically, if $|C_{31}| \neq 1$, take

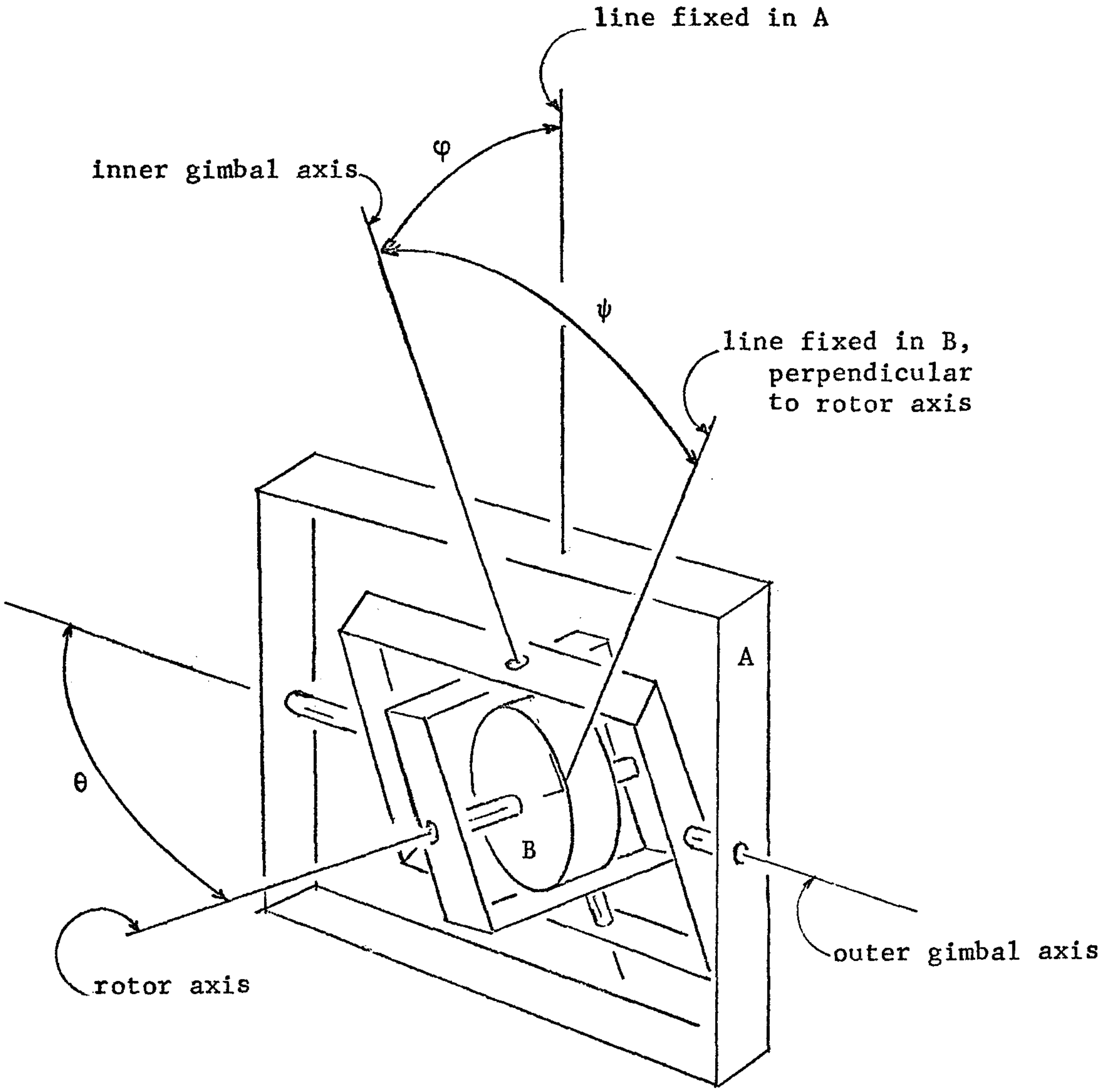
line fixed in A
inner gimbal axis
φ
ψ
line fixed in B,
perpendicular
to rotor axis
A
θ
B
rotor axis
outer gimbal axis

Figure 1.7.1

$$\theta_2 = \sin^{-1}(-C_{31}) \ , \qquad -\frac{\pi}{2} < \theta_2 < \frac{\pi}{2} \tag{2}$$

Next, after evaluating c_2 , define α as

$$\alpha \triangleq \sin^{-1}(C_{32}/c_2) \ , \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \tag{3}$$

and let

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{33} \geq 0 \\ \pi - \alpha & \text{if } C_{33} < 0 \end{cases} \tag{4}$$

Similarly, define β as

$$\beta \triangleq \sin^{-1}(C_{21}/c_2) \ , \qquad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \tag{5}$$

and take

$$\theta_3 = \begin{cases} \beta & \text{if } C_{11} \geq 0 \\ \pi - \beta & \text{if } C_{11} < 0 \end{cases} \tag{6}$$

If $|C_{31}| = 1$, take

$$\theta_2 = \begin{cases} -\frac{\pi}{2} & \text{if } C_{31} = 1 \\ \frac{\pi}{2} & \text{if } C_{31} = -1 \end{cases} \tag{7}$$

and, after defining α as

$$\alpha \triangleq \sin^{-1}(-C_{23}) , \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \tag{8}$$

let

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{22} \geq 0 \\ \pi - \alpha & \text{if } C_{22} < 0 \end{cases} \tag{9}$$

and

$$\theta_3 = 0 \tag{10}$$

In other words, two rotations suffice in this case.

A second method for accomplishing the same objective is to subject B successively to a $\underline{b}_1$ rotation of amount θ_1 , a $\underline{b}_2$ rotation of amount θ_2 , and a $\underline{b}_3$ rotation of amount θ_3 . The matrix C relating $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, to $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, as in Eq. (1.2.2) subsequent to the last rotation is then given by

$$C = \begin{bmatrix} c_2c_3 & -c_2s_3 & s_2 \\ s_1s_2c_3 + s_3c_1 & -s_1s_2s_3 + c_3c_1 & -s_1c_2 \\ -c_1c_2c_3 + s_cs_1 & c_1s_2s_3 + c_3s_1 & c_1c_2 \end{bmatrix} \tag{11}$$

and, if $|C_{13}| \neq 1$, suitable values of θ_1 , θ_2 , and θ_3 are obtained by taking

$$\theta_2 = \sin^{-1}(C_{13}) \ , \qquad -\frac{\pi}{2} < \theta_2 < \frac{\pi}{2} \tag{12}$$

$$\alpha \triangleq \sin^{-1}(-C_{23}/c_2) \ , \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \tag{13}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{33} \geq 0 \\ \pi - \alpha & \text{if } C_{33} < 0 \end{cases} \tag{14}$$

$$\beta \triangleq \sin^{-1} (-C_{12}/c_2) \ , \qquad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \tag{15}$$

$$\theta_3 = \begin{cases} \beta & \text{if } C_{11} \geq 0 \\ \pi - \beta & \text{if } C_{11} < 0 \end{cases} \tag{16}$$

whereas, if $|C_{13}| = 1$, one may let

$$\theta_2 = \begin{cases} \frac{\pi}{2} & \text{if } C_{13} = 1 \\ -\frac{\pi}{2} & \text{if } C_{13} = -1 \end{cases} \tag{17}$$

$$\alpha \triangleq \sin^{-1}(C_{32}) \ , \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \tag{18}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{22} \geq 0 \\ \pi - \alpha & \text{if } C_{22} < 0 \end{cases} \tag{19}$$

$$\theta_3 = 0 \tag{20}$$

so that, once again, only two rotations are required.

The physical difference btween these two procedures for bringing B into a desired orientation in A is that the first involves unit vectors fixed in the reference frame, whereas the second brings unit vectors fixed in the body into play. What the two methods have in common is that three distinct unit vectors are employed in both cases.

It is also possible to bring B into an arbitrary orientation relative to A by performing three successive rotations which involve only two distinct unit vectors, and these vectors may be fixed either in the reference frame or in the body. Specifically, if B is subjected successively to an $\underline{a}_1$ rotation of amount θ_1, an $\underline{a}_2$ rotation of amount θ_2, and again an $\underline{a}_1$ rotation, but this time of amount θ_3, then

$$C = \begin{bmatrix} c_2 & s_1 s_2 & c_1 s_2 \\ s_2 s_3 & -s_1 c_2 s_3 + c_3 c_1 & -c_1 c_2 s_3 - c_3 s_1 \\ -s_2 c_3 & s_1 c_2 c_3 + s_3 c_1 & c_1 c_2 c_3 - s_3 s_1 \end{bmatrix} \tag{21}$$

and, if $|C_{11}| \neq 1$, one can take

$$\theta_2 = \cos^{-1}(C_{11}) \,, \qquad 0 < \theta_2 < \pi \tag{22}$$

$$\alpha \triangleq \sin^{-1}(C_{12}/s_2) \,, \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \tag{23}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{13} \geq 0 \\ \pi - \alpha & \text{if } C_{13} < 0 \end{cases} \tag{24}$$

$$\beta \triangleq \sin^{-1}(C_{21}/s_2) \ , \qquad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \tag{25}$$

$$\theta_3 = \begin{cases} \beta & \text{if } C_{31} < 0 \\ \pi - \beta & \text{if } C_{31} \ge 0 \end{cases} \tag{26}$$

while, if $|C_{11}| = 1$, one may let

$$\theta_2 = \begin{cases} 0 \text{ if } C_{11} = 1 \\ \pi \text{ if } C_{11} = -1 \end{cases} \tag{27}$$

$$\alpha \triangleq \sin^{-1}(-C_{23}) \ , \ -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2} \tag{28}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{22} \geqslant 0 \\ \pi - \alpha & \text{if } C_{22} < 0 \end{cases} \tag{29}$$

$$\theta_3 = 0 \tag{30}$$

Finally, if the successive rotations are a $\underline{b}_1$ rotation of amount θ_1 , a $\underline{b}_2$ rotation of amount θ_2, and again a $\underline{b}_1$ rotation, but this time of amount θ_3 , then

$$C = \begin{bmatrix} c_2 & s_2 s_3 & s_2 c_3 \\ s_1 s_2 & -s_1 c_2 s_3 + c_3 c_1 & -s_1 c_2 c_3 - s_3 c_1 \\ -c_1 s_2 & c_1 c_2 s_3 + c_3 s_1 & c_1 c_2 c_3 - s_3 s_1 \end{bmatrix} \tag{31}$$

and, if $|C_{11}| \neq 1$, θ_1 , θ_2 , and θ_3 may be found by taking

$$\theta_2 = \cos^{-1}(C_{11}) , \qquad 0 < \theta_2 < \pi \tag{32}$$

$$\alpha \triangleq \sin^{-1}(C_{21}/s_2) , \qquad -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2} \tag{33}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{31} < 0 \\ \pi - \alpha & \text{if } C_{31} \geqslant 0 \end{cases} \tag{34}$$

$$\beta \triangleq \sin^{-1}(C_{12}/s_2) , \qquad -\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2} \tag{35}$$

$$\theta_3 = \begin{cases} \beta & \text{if } C_{13} \geqslant 0 \\ \pi - \beta & \text{if } C_{13} < 0 \end{cases} \tag{36}$$

while, if $|C_{11}| = 1$, one can use

$$\theta_2 = \begin{cases} 0 \text{ if } C_{11} = 1 \\ \pi \text{ if } C_{11} = -1 \end{cases} \tag{37}$$

$$\alpha \triangleq \sin^{-1}(C_{32}) , \qquad -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2} \tag{38}$$

$$\theta_1 = \begin{cases} \alpha & \text{if } C_{22} \geqslant 0 \\ \pi - \alpha & \text{if } C_{22} < 0 \end{cases} \tag{39}$$

$$\theta_3 = 0 \tag{40}$$

The matrices in Eqs. (1) and (11) are intimately related to each other: Either one may be obtained from the other by replacing θ_i with $-\theta_i$ $(i = 1,2,3)$ and transposing. The matrices in Eqs. (21) and (31) are related similarly. These facts have the following physical significance, as may be verified by using Eq. (1.6.4): If B is subjected successively to an $\underline{a}_1$, an $\underline{a}_2$, and an $\underline{a}_3$ rotation of amount θ_1, θ_2, and θ_3, respectively, then B can be brought back into its original orientation in A by next subjecting B to successive $-\underline{b}_1$, $-\underline{b}_2$, and $-\underline{b}_3$ rotations of amounts θ_1, θ_2, and θ_3, respectively. Similarly, employing only four unit vectors, one can subject B to successive rotations characterized by $\theta_1\underline{a}_1$, $\theta_2\underline{a}_2$, $\theta_3\underline{a}_1$, $-\theta_1\underline{b}_1$, $-\theta_2\underline{b}_2$, and $-\theta_3\underline{b}_1$ without producing any ultimate change in the orientation of B in A. Furthermore, it does not matter whether the rotations involving unit vectors fixed in A are preceded or followed by those involving unit vectors fixed in B; that is, the sequences of successive rotations represented by $\theta_1\underline{b}_1$, $\theta_2\underline{b}_2$, $\theta_3\underline{b}_3$, $-\theta_1\underline{a}_1$, $-\theta_2\underline{a}_2$, $-{}_3\underline{a}_3$ and by $\theta_1\underline{b}_1$, $\theta_2\underline{b}_2$, $\theta_3\underline{b}_1$, $-\theta_1\underline{a}_1$, $-\theta_2\underline{a}_2$, $-\theta_3\underline{a}_1$ also have no net effect on the orientation of B in A.

To indicate which set of three angles one is using, one can speak of "space-three-vector angles" in connection with Eqs. (1) - (10), "body-three-vector angles" for Eqs. (11) - (20), "space-two-vector angles" for Eqs. (21) - (30), and "body-two-vector angles" for Eqs. (31) - (40); and this terminology remains meaningful even when the angles and unit vectors employed are denoted by symbols other than those used in Eqs. (1) - (40). Moreover, once one has identified three angles in this way, one can always find appropriate replacements for

Eqs. (11), (21), (31), or (41) by direct use of these equations. Suppose, for example, that $\underline{x}$, $\underline{y}$, $\underline{z}$ and $\underline{\xi}$, $\underline{\eta}$, $\underline{\zeta}$ are dextral sets of orthogonal unit vectors fixed in a reference frame A and in a rigid body B , respectively; that $\underline{x} = \underline{\xi}$, $\underline{y} = \underline{\eta}$, and $\underline{z} = \underline{\zeta}$ initially ; that B is subjected, successively, to a $\underline{z}$ rotation of amount γ , a $\underline{y}$ rotation of amount β , and an $\underline{x}$ rotation of amount α ; and that it is required to find the elements L_{ij} $(i,j = 1,2,3)$ of the matrix L such that, subsequent to the last rotation,

$$[\underline{\xi}\ \underline{\eta}\ \underline{\zeta}] = [\underline{x}\ \underline{y}\ \underline{z}]\ L$$

Then, recognizing α , β , and γ as space-three-vector angles, one can introduce $\underline{a}_i$, $\underline{b}_i$, and θ_i $(i = 1,2,3)$ as

$$\underline{a}_1 \triangleq \underline{z} , \qquad \underline{a}_2 \triangleq \underline{y} , \qquad \underline{a}_3 \triangleq - \underline{x}$$

$$\underline{b}_1 \triangleq \underline{\zeta} , \qquad \underline{b}_2 \triangleq \underline{\eta} , \qquad \underline{b}_3 \triangleq - \underline{\xi}$$

and

$$\theta_1 \triangleq \gamma , \qquad \theta_2 \triangleq \beta , \qquad \theta_3 \triangleq -\alpha$$

in which case the given sequence of rotation is represented by $\theta_1\underline{a}_1$, $\theta_2\underline{a}_2$, and $\theta_3\underline{a}_3$; and L_{ij} can then be found by referring to Eq. (1) to express the scalar product associated with L_{ij} in terms of α , β , and γ . For instance,

$$L_{21} = \underline{y} \cdot \underline{\xi} = \underline{a}_2 \cdot (-\underline{b}_3)$$

$$\underset{(1.2.1)}{=} - C_{23} \underset{(1)}{=} -c_1 s_2 s_3 + c_3 s_1$$

$$= \cos\gamma \sin\beta \sin\alpha + \cos\alpha \sin\gamma$$

<u>Derivations</u>: To establish the validity of Eq. (1), one may use Eq. (1.6.15), forming ${}^{A}C^{\overline{B}}$, ${}^{\overline{B}}C^{\overline{\overline{B}}}$, and ${}^{\overline{\overline{B}}}C^{B}$ with the aid of Eq. (1.2.35) and Eqs. (1.2.23) - (1.2.31). Specifically, to deal with the $\underline{a}_1$ rotation, let

$$^{A}C^{\overline{B}} \underset{(1.2.35)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} \tag{a}$$

Next, to construct a matrix ${}^{\overline{B}}C^{\overline{\overline{B}}}$ that characterizes the $\underline{a}_2$ rotation, let ${}^{A}v$ and ${}^{\overline{B}}v$ denote row matrices whose elements are $\underline{a}_2 \cdot \underline{a}_i$ $(i = i,2,3)$ and $\underline{a}_2 \cdot \overline{\underline{b}}_i$ $(i = 1,2,3)$, respectively. Then

$$^{A}v = [0\ 1\ 0] \tag{b}$$

$$^{\overline{B}}v \underset{(1.2.9)}{=} {}^{A}v\ {}^{A}C^{\overline{B}} \underset{(a,b)}{=} [0\ c_1\ -s_1] \tag{c}$$

and, from Eqs. (1.2.23) - (1.2.31), with $\lambda_1 = 0$, $\lambda_2 = c_1$, $\lambda_3 = -s_1$, and $\theta = \theta_2$,

$$
{}^{\bar{B}}C^{\bar{\bar{B}}} = \begin{bmatrix} c_2 & s_1s_2 & c_1s_2 \\ -s_1s_2 & 1+s_1{}^2(c_2-1) & s_1c_1(c_2-1) \\ -c_1s_2 & s_1c_1(c_2-1) & 1+c_1{}^2(c_2-1) \end{bmatrix} \tag{d}
$$

A matrix ${}^{A}C^{\bar{\bar{B}}}$ associated with a simple rotation that is equivalent to the first two rotations is now given by

$$
{}^{A}C^{\bar{\bar{B}}} \underset{(1.6.4)}{=} {}^{A}C^{\bar{B}}\,{}^{\bar{B}}C^{\bar{\bar{B}}} \underset{(a,d)}{=} \begin{bmatrix} c_2 & s_1s_2 & c_1s_2 \\ 0 & c_1 & -s_1 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix} \tag{e}
$$

and, to resolve $\underline{a}_3$ into components required for the construction of a matrix ${}^{\bar{\bar{B}}}C^{B}$, one may use Eq. (1.2.9) to obtain

$$
[0\ 0\ 1]\ {}^{A}C^{\bar{\bar{B}}} \underset{(e)}{=} [-s_2\ s_1c_2\ c_1c_2]
$$

after which Eqs. (1.2.23) - (1.2.31) yield

$$
{}^{\bar{\bar{B}}}C^{B} = \begin{bmatrix} s_2{}^2+c_2{}^2c_3 & -c_2[c_1s_3+s_1s_2(1-c_3)] & c_2[s_1s_3-c_1s_2(1-c_3)] \\ c_2[c_1s_3+s_1s_2(1-c_3)] & c_3(1-s_1{}^2c_2{}^2)+s_1{}^2c_2{}^2 & s_2s_3+s_1c_1c_2{}^2(1-c_3) \\ -c_2[s_1s_3+c_1s_2(1-c_3)] & -s_2s_3+s_1c_1c_2{}^2)1-c_3) & 1-(s_2{}^2+s_1{}^2c_2{}^2)(1-c_3) \end{bmatrix} \tag{f}
$$

and substitution from Eqs. (a), (d), and (f) into Eq. (1.6.15) leads directly to Eq. (1).

Eq. (11) may be derived by using Eq. (1.6.15) with

$$ {}^{A}C^{\bar{B}} \underset{(1.2.35)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} $$

$$ {}^{\bar{B}}C^{\bar{\bar{B}}} \underset{(1.2.36)}{=} \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} $$

and

$$ {}^{\bar{\bar{B}}}C^{B} \underset{(1.2.37)}{=} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Eqs. (2) - (10) and (12) - (20) are immediate consequences of Eq. (1) and Eq. (11), respectively; and Eqs. (21) - (40) can be generated by procedures similar to those employed in the derivation of Eqs. (1) - (20).

<u>Example</u>: If unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are introduced as shown in Fig. 1.7.2, and the angles ϕ , θ , and ψ shown in Fig. 1.7.1 are re-named θ_1 , θ_2 , and θ_3 respectively, then θ_1 , θ_2 , and θ_3 are body-two-vector angles such that Eqs. (31) - (40) can be used to discuss motions of B in A . However, as will be seen later, it is undesirable to use these angles when dealing with motions during which the rotor axis becomes coincident, or even nearly coincident, with the outer gimbal axis. (Coincidence of these two axes is referred to as "gimbal lock".) Therefore, it may be convenient to employ in the course of one analysis two modes of description of the

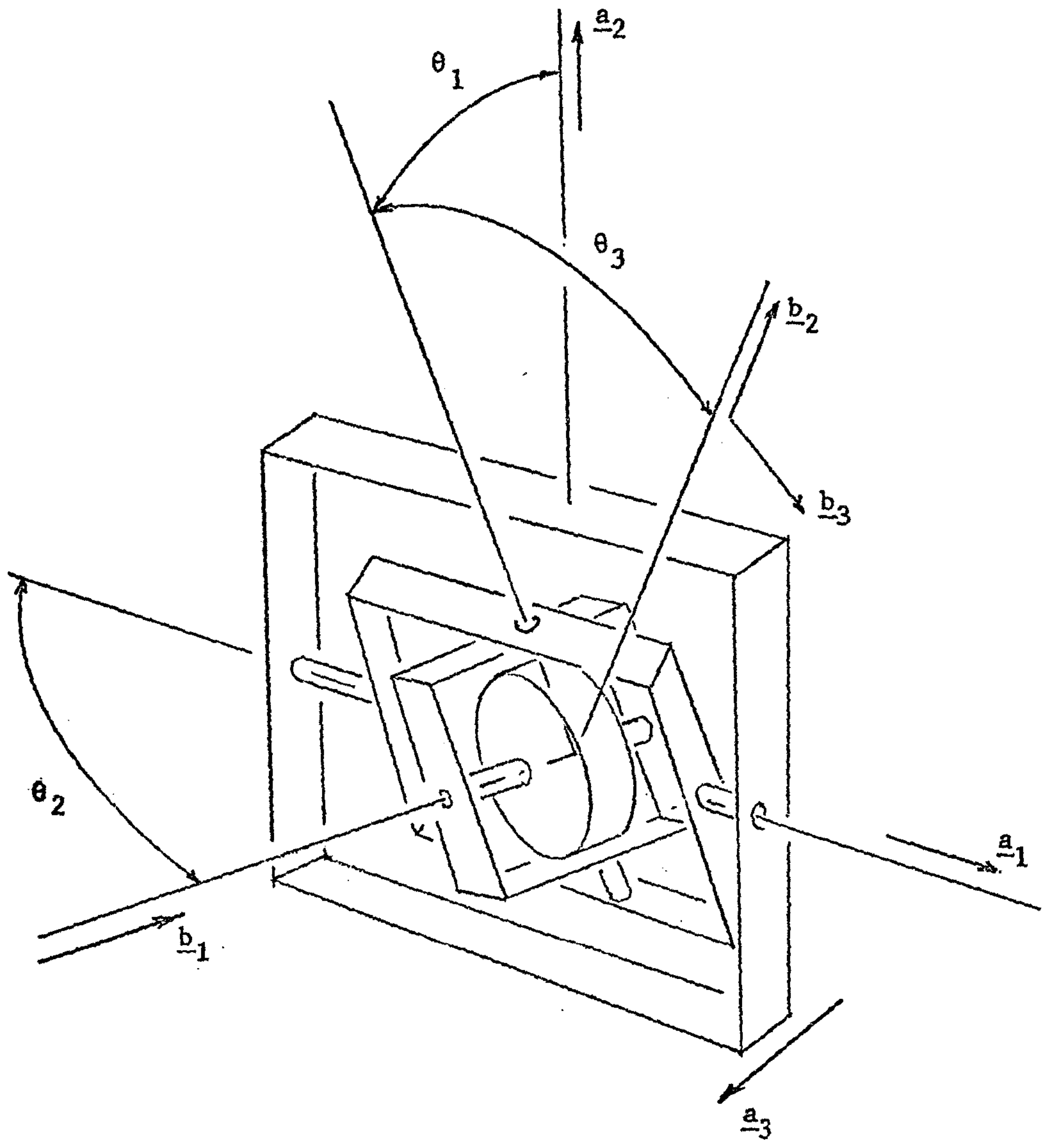
θ_1
$\underset{\sim}{a}_2$
θ_3
$\underset{\sim}{b}_2$
$\underset{\sim}{b}_3$
θ_2
$\underset{\sim}{a}_1$
$\underset{\sim}{b}_1$
$\underset{\sim}{a}_3$

Figure 1.7.2

orientation of B in A , switching from one to the other whenever θ_2 acquires a value lying in a previously designated range. The following sort of question can then arise: If ϕ_1 , ϕ_2 , and ϕ_3 are the space-three-vector angles associated with $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, what are the values of these angles corresponding to $\theta_1 = 30°$, $\theta_2 = 45°$, $\theta_3 = 60°$?

Inspection of Eqs. (2) - (6) shows that the elements of C required for the evaluation of ϕ_1 , ϕ_2 , and ϕ_3 are C_{31} , C_{32} , C_{33} , C_{21} , and C_{11}. From Eq. (31),

$$C_{31} = -\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = -0.613$$

$$C_{32} = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{1}{2} = 0.780$$

$$C_{33} = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\;\frac{1}{2} - \frac{\sqrt{3}}{2}\frac{1}{2} = -0.127$$

$$C_{21} = \frac{1}{2}\frac{\sqrt{2}}{2} = 0.354$$

$$C_{11} = \frac{\sqrt{2}}{2} = 0.707$$

Hence

$$\phi_2 \underset{(2)}{=} \sin^{-1}(0.613) = 37.8°$$

$$\alpha \underset{(3)}{=} \sin^{-1}(0.780/0.791) = 80.0°$$

1.7

$$\phi_1 \underset{(4)}{=} 100.0°$$

$$\beta \underset{(5)}{=} 26.5°$$

$$\phi_3 \underset{(6)}{=} 26.5°$$

1.8 Small rotations

When a simple rotation (see Sec. 1.1) is small in the sense that second and higher powers of θ play a negligible role in an analysis involving the rotation, a number of the relationships discussed heretofore can be replaced with simpler ones. For example, Eqs. (1.1.1) and (1.1.2) yield respectively

$$\underline{b} = \underline{a} - \underline{a} \times \underline{\lambda}\theta \tag{1}$$

and

$$\underline{C} = \underline{U} - \underline{U} \times \underline{\lambda}\theta \tag{2}$$

while Eqs. (1.3.1), (1.3.3) and (1.4.1) give way to

$$\underline{\varepsilon} = \frac{1}{2}\,\underline{\lambda}\theta \tag{3}$$

$$\varepsilon_4 = 1 \tag{4}$$

and

$$\underline{\rho} = \frac{1}{2}\,\underline{\lambda}\theta \tag{5}$$

which shows that, to the order of approximation under consideration, the Rodrigues vector is indistinguishable from the Euler vector.

As will be seen presently, analytical descriptions of small rotations frequently involve skew-symmetric matrices. In dealing with these, it is convenient to establish the notational convention that the symbol obtained by placing a tilde over a letter, say q, denotes a skew-symmetric matrix whose off-diagonal elements have values denoted by $\pm q_i$ $(i = 1,2,3)$, these elements being arranged as follows:

$$\tilde{q} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{6}$$

Using this convention, one can express the results obtained by neglecting second and higher powers of θ in Eqs. (1.2.23) - (1.2.31) as

$$C = U + \tilde{\lambda}\theta \tag{7}$$

Similarly, Eqs. (1.3.6) - (1.3.14) yield

$$C = U + 2\tilde{\varepsilon} \tag{8}$$

Considering two successive small rotations, suppose that ${}^{A}C^{\bar{B}}$ and ${}^{\bar{B}}C^{B}$ are direction cosine matrices characterizing the first and second such rotation as in Sec. 1.6. Then, instead of using Eq. (1.6.4), one can express the direction cosine matrix ${}^{A}C^{B}$ associated with a single equivalent small rotation as

$${}^{A}C^{B} = U + ({}^{A}C^{\bar{B}} + {}^{\bar{B}}C^{B} - 2U) \tag{9}$$

Similarly, for three small rotations Eq. (1.6.15) leads to

$$ {}^{A}C^{B} = U + \left({}^{A}C^{\bar{B}} + {}^{\bar{B}}C^{\bar{\bar{B}}} + {}^{\bar{\bar{B}}}C^{B} - 3U \right) \tag{10} $$

Rodrigues vectors ${}^{A}\underline{\rho}^{\bar{B}}$, ${}^{\bar{B}}\underline{\rho}^{B}$, and ${}^{A}\underline{\rho}^{B}$ associated respectively with a first, a second, and an equivalent single small rotation satisfy the equation

$$ {}^{A}\underline{\rho}^{B} = {}^{A}\underline{\rho}^{\bar{B}} + {}^{\bar{B}}\underline{\rho}^{B} \tag{11} $$

Both the relationship and Eq. (9) show that the final orientation of B in A is independent of the order in which two small successive rotations are performed.

Finally when θ_1 , θ_2 , and θ_3 in Eqs. (1.7.1) - (1.7.40) are small in the sense that terms of second or higher degree in these quantities are negligible, then Eqs. (1.7.1) and (1.7.11) each yield (see Eq. (6) for the meaning of $\tilde{\theta}$)

$$ C = U + \tilde{\theta} \tag{12} $$

showing that it is immaterial whether one uses space-three-vector angles or body-three-vector angles under these circumstances. The relationship corresponding to Eq. (12) for two-vector angles, namely

$$ C \underset{(1.7.21)}{=} U + \begin{bmatrix} 0 & 0 & \theta_2 \\ 0 & 0 & -(\theta_3 + \theta_1) \\ -\theta & (\theta_3 + \theta_1) & 0 \end{bmatrix} \tag{13} $$

is less useful because this equation cannot be solved uniquely for θ_1 and θ_3 as functions of C_{ij} $(i,j = 1,2,3)$.

Derivations: Eq. (1) follows from Eq. (1.1.1) when $\sin\theta$ is replaced with θ and $\cos\theta$ with unity. The same substitution in Eq. (1.1.2) leads to Eq. (2). Eqs. (3) and (4) are obtained by replacing $\sin(\theta/2)$ with $\theta/2$ and $\cos(\theta/2)$ with unity in Eqs. (1.3.1) and (1.3.3), and Eq. (5) follows from Eq. (1.4.1) when $\tan(\theta/2)$ is replaced with $\theta/2$.

Eq. (7) follows directly from Eqs. (1.2.23) - (1.2.31), and Eq. (8) results from dropping terms of second degree in ε_1, ε_2, and/or ε_3 when forming C_{ij} in accordance with Eq. (1.3.6) - (1.3.14), which is justified in view of Eqs. (3) and (1.3.2).

To establish the validity of Eq. (9) one may proceed as follows: Using Eq. (7), one can express the matrices ${}^{A}C^{\bar{B}}$ and ${}^{\bar{B}}C^{\bar{\bar{B}}}$ introduced in Sec. 1.6 as

$$ {}^{A}C^{\bar{B}} = U + \tilde{\lambda}\theta $$

and

$$ {}^{\bar{B}}C^{\bar{\bar{B}}} = U + \tilde{\mu}\phi $$

where θ and ϕ are respectively the radian measures of the first and of the second small rotation, and $\tilde{\lambda}$ and $\tilde{\mu}$ characterize the associated axes of rotation. Substituting into Eq. (1.6.4), and dropping the product $\tilde{\lambda}\tilde{\mu}\theta\phi$, one then obtains

$$
\begin{aligned}
{}^{A}C^{B} &= U + \tilde{\lambda}\theta + \tilde{\mu}\phi \\
&= U + (U + \tilde{\lambda}\theta + U + \tilde{\mu}\phi - 2U) \\
&= U + ({}^{A}C^{\overline{B}} + {}^{\overline{B}}C^{\overline{\overline{B}}} - 2U)
\end{aligned}
$$

A similar procedure leads to Eq. (10).

Finally, Eq. (11) may be obtained by using Eq. (5) in conjunction with Eq. (1.6.5), and Eqs. (12) and (13) result from linearizing in θ_i $(i = 1,2,3)$ in Eqs. (1.7.11) and (1.7.21), respectively, and using the convention established in Eq. (6).

Example: In Fig. 1.8.1, $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ form a dextral set of orthogonal unit vectors, with $\underline{a}_1$ and $\underline{a}_2$ parallel to edges of a rectangular plate B ; and X and Y designate lines perpendicular to $\underline{a}_1$ and $\underline{a}_2$, respectively. When B is subjected, successively, to a rotation of amount 0.01 rad. about X and a rotation of amount 0.02 rad. about Y , the sense of each rotation being that indicated in the sketch, point P traverses a distance d . This distance is to be determined on the assumption that the two rotations can be regarded as small.

If $\underline{a}$ designates the position vector of point P relative to point O before the rotations are performed, and $\underline{b}$ the rotation vector of P relative to O subsequent to the second rotation, then

$$d = |\underline{b} - \underline{a}| \tag{a}$$

with

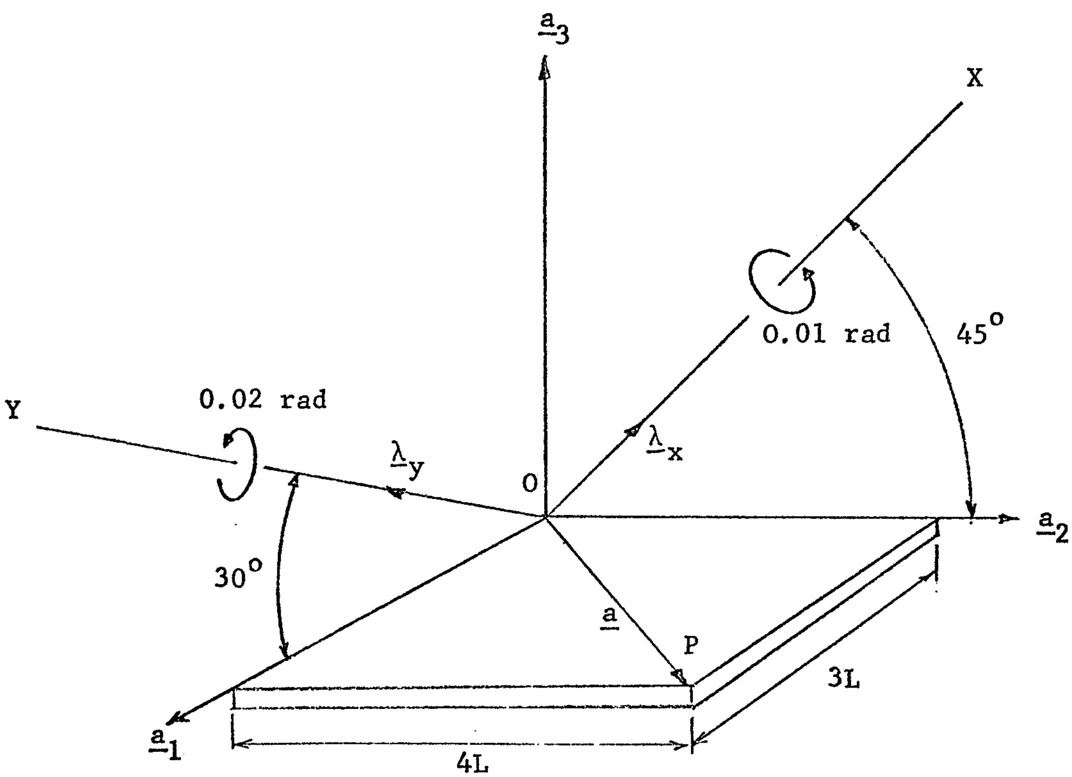
$\underline{a}_3$
X
0.01 rad
45^o
Y
0.02 rad
$\underline{\lambda}_y$
$\underline{\lambda}_x$
O
$\underline{a}_2$
30^o
$\underline{a}$
P
3L
$\underline{a}_1$
4L

Figure 1.8.1

$$\underline{a} = L(3\underline{a}_1 + 4\underline{a}_2) \tag{b}$$

and

$$\underline{b} \underset{(1)}{=} \underline{a} - \underline{a} \times \underline{\lambda}\theta \tag{c}$$

where $\underline{\lambda}$ and θ are respectively a unit vector and the radian measure of an angle associated with a single rotation that is equivalent to the two given rotations. To determine the product $\underline{\lambda}\theta$, let $\underline{\rho}$ be the Rodrigues vector for this equivalent rotation, in which case

$$\underline{\lambda}\theta \underset{(5)}{=} 2\underline{\rho} \tag{d}$$

and refer to Eq. (11) to express $\underline{\rho}$ as

$$\underline{\rho} = \underline{\rho}_x + \underline{\rho}_y \underset{(5)}{=} \frac{0.01}{2}\,\underline{\lambda}_x + \frac{0.02}{2}\,\underline{\lambda}_y \tag{e}$$

where $\underline{\lambda}_x$ and $\underline{\lambda}_y$ are unit vectors directed as shown in Fig. 1.8.1; that is,

$$\underline{\lambda}_x = \frac{1}{\sqrt{2}}\,(\underline{a}_2 + \underline{a}_3) \tag{f}$$

and

$$\underline{\lambda}_y = \frac{1}{2}\,(\sqrt{3}\underline{a}_1 + \underline{a}_3) \tag{g}$$

then

$$\underline{\lambda}\theta \underset{(d-e)}{=} 0.01\underline{\lambda}_x + 0.02\underline{\lambda}_y$$

$$\underset{(f,g)}{=} \frac{0.01}{\sqrt{2}} [\sqrt{6}\underline{a}_1 + \underline{a}_2 + (1 + \sqrt{2})\underline{a}_3] \qquad \text{(h)}$$

$$\underline{b} - \underline{a} \underset{(c)}{=} -\underline{a} \times (\underline{\lambda}\theta)$$

$$\underset{(b,h)}{=} \frac{0.01}{\sqrt{2}} [-4(1 + \sqrt{2})\underline{a}_1 + 3(1 + \sqrt{2})\underline{a}_2 + (4\sqrt{6} - 3)\underline{a}_3]L$$

and

$$d \underset{(a,i)}{=} L\left[\frac{16(1 + \sqrt{2})^2 + 9(1 + \sqrt{2})^2 + (4\sqrt{6} - 3)^2}{20000}\right]^{\frac{1}{2}} = 0.098L$$

1.9 Screw Motion

If P_1 and P_2 are points fixed in a reference frame A , and a point P is moved from P_1 to P_2 , then P is said to experience a displacement in A , and the position vector of P_2 relative to P_1 is called a displacement vector of P in A .

When points of a rigid body B experience displacements in a reference frame A , one speaks of a displacement of B in A ; and a displacement of B in A is called a translation of B in A if the displacement vectors of all points of B in A are equal to each other.

Every displacement of a rigid body B in a reference frame A can be produced by subjecting B successively to a translation in which a basepoint P of B , chosen arbitrarily, is brought from its original to its terminal position, and a simple rotation (see Sec. 1.1) during which P remains fixed in A . The Rodrigues vector (see Sec. 1.4) for the simple rotation is independent of the choice of basepoint, whereas the displacement vector of the basepoint depends on this choice. When the displacement vector of the basepoint is parallel to the Rodrigues vector for the rotation, the displacement under consideration is said to be produced by means of a screw motion.

Every displacement of a rigid body B in a reference frame A can be produced by means of a screw motion. In other words, one can always find a basepoint whose displacement vector is parallel to the Rodrigues vector for the simple rotation associated with a displacement of B in A . In fact, there exist infinitely many such basepoints, all lying on a straight line that is parallel to the Rodigues vector and bears the name <u>screw axis</u>; and the displacement vectors of all points of B lying on the screw axis are equal to each other and can, therefore, be characterized by a single vector, called the <u>screw translation vector</u>. Moreover, the magnitude of the screw translation vector is either smaller than or equal to the magnitude of the displacement vector of any basepoint not lying on the screw axis.

If $\underline{\delta}$ is the displacement vector of an arbitrary basepoint P , $\underline{\rho}$ is the Rodrigues vector for the rotation associated with a displacement of B in A, and P^* is a point of B lying on the screw axis (see Fig. 1.9.1), then the position vector $\underline{a}^*$ of P^* relative to P prior to the displacement of B in A satisfies the equation

$$\underline{a}^* = \frac{\underline{\rho} \times \underline{\delta} + (\underline{\rho} \times \underline{\delta}) \times \underline{\rho}}{2\underline{\rho}^2} + \mu\underline{\rho} \tag{1}$$

where μ depends on the choice of P^* ; and the screw translation vector $\underline{\delta}^*$ is given by

$$\underline{\delta}^* = \frac{\underline{\rho} \cdot \underline{\delta}}{\underline{\rho}^2}\,\underline{\rho} \tag{2}$$

<u>Derivation</u>: If both P^* and P are points of B selected arbitrarily, and $\underline{a}^*$ is the position vector of P^* relative to P prior to the displacement of B in A , while $\underline{b}^*$ is the position vector of P^* relative to P subsequent to this displacement, then the displacement vector $\underline{\delta}^*$ of P^* can be expressed as (see Fig. 1.9.1)

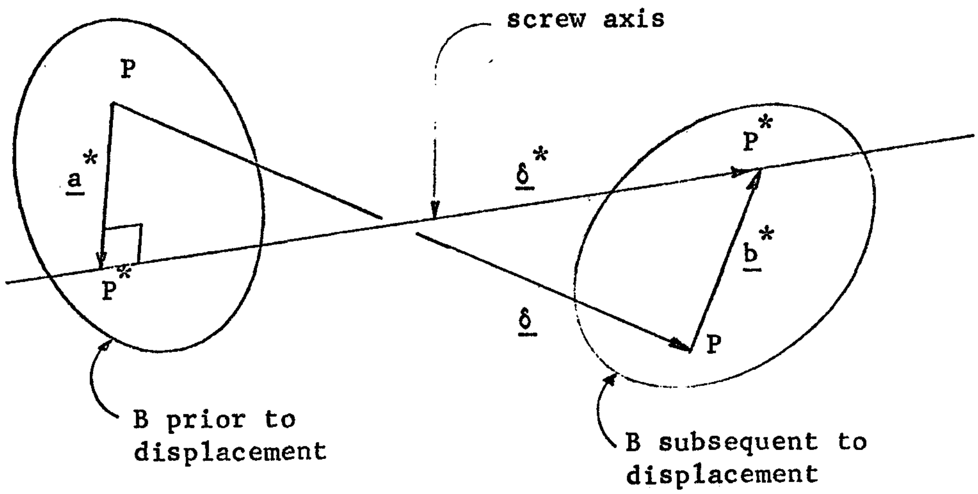
screw axis
P
$\underline{a}^*$
P^*
B prior to displacement
$\underline{\delta}^*$
P^*
$\underline{b}^*$
$\underline{\delta}$
P
B subsequent to displacement

Figure 1.9.1

$$\underline{\delta}^* = \underline{\delta} + \underline{b}^* - \underline{a}^* \tag{a}$$

where $\underline{\delta}$ is the displacement vector of P ; and

$$\underline{b}^* - \underline{a}^* \underset{(1.4.5)}{=} \underline{\rho} \times (\underline{a}^* + \underline{b}^*) \tag{b}$$

so that

$$\underline{\delta}^* \underset{(a,b)}{=} \underline{\delta} + \underline{\rho} \times (\underline{a}^* + \underline{b}^*) \tag{c}$$

Hence, if P^* is to be chosen such that $\underline{\delta}^*$ be parallel to $\underline{\rho}$, in which case $\underline{\rho} \times \underline{\delta}^*$ is equalto zero, then $\underline{a}^*$ must satisfy the equation

$$\underline{\rho} \times \underline{\delta} + \underline{\rho} \times [\underline{\rho} \times (\underline{a}^* + \underline{b}^*)] \underset{(c)}{=} 0 \tag{d}$$

or, equivalently,

$$\underline{\rho} \times \underline{\delta} + \underline{\rho} \cdot (\underline{a}^* + \underline{b}^*)\, \underline{\rho} - \underline{\rho}^2(\underline{a}^* + \underline{b}^*) = 0 \tag{e}$$

so that

$$\underline{a}^* + \underline{b}^* \underset{(e)}{=} \frac{\underline{\rho} \times \underline{\delta}}{\underline{\rho}^2} + \frac{\underline{\rho} \cdot (\underline{a}^* + \underline{b}^*)}{\underline{\rho}^2}\, \underline{\rho} \tag{f}$$

and

$$\underline{\delta}^* \underset{(c,f)}{=} \underline{\delta} + \underline{\rho} \times \frac{\underline{\rho} \times \underline{\delta}}{\underline{\rho}^2} = \frac{\underline{\rho} \cdot \underline{\delta}}{\underline{\rho}^2} \underline{\rho}$$

in agreement with Eq. (2). As for Eq. (1), one may solve the equation

$$\underline{b}^* - \underline{a}^* \underset{(b,f)}{=} \underline{\rho} \times \frac{\underline{\rho} \times \underline{\delta}}{\underline{\rho}^2}$$

for $\underline{b}^*$, substitute the result into Eq. (f), obtaining

$$\underline{a}^* = \frac{\underline{\rho} \times \underline{\delta} + (\underline{\rho} \times \underline{\delta}) \times \underline{\rho}}{2\underline{\rho}^2} + \frac{\underline{\rho} \cdot \underline{a}^*}{\underline{\rho}^2}$$

and then simply define μ as $\underline{\rho} \cdot \underline{a}^* \underline{\rho}/\underline{\rho}^2$. Moreover, this equation shows that the locus of basepoints whose displacement vectors are parallel to $\underline{\rho}$ is a straight line parallel to $\underline{\rho}$.

The contention that the magnitude of the screw translation vector is either smaller than or equal to the magnitude of the displacement vector of any point not lying on the screw axis is based on the observation that

$$\underline{\delta}^* \underset{(2)}{=} \frac{\underline{\rho} \cdot \underline{\delta}}{\underline{\rho}^2} \underline{\rho} = \frac{\underline{\rho} \cdot \underline{\delta}}{\underline{\rho}}$$

$$= \left| \frac{\underline{\rho}}{|\underline{\rho}|} \cdot \underline{\delta} \right| \le |\underline{\delta}|$$

Example: The Example in Sec. 1.3 dealt with a displacement of the triangle ABC shown in Fig. 1.9.2. The displacement in question was one that could be produced by performing a translation of the triangle during which point A is brought to A' , and following this with a rotation during which point A remains fixed at A' ; and the Euler vector $\underline{\varepsilon}$ and Euler parameter ϵ_4 for the rotation were found to be

$$\underline{\epsilon} = \tfrac{1}{2}(\underline{a}_1 - \underline{a}_2 - \underline{a}_3)$$

and

$$\epsilon_4 = \tfrac{1}{2}$$

where $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are unit vectors directed as shown in Fig. 1.9.2. To determine how the triangle can be brought into the same ultimate position by means of a screw motion, form $\underline{\rho}$ by reference to Eqs. (1.4.2) and (1.4.3), obtaining

$$\underline{\rho} = \underline{a}_1 - \underline{a}_2 - \underline{a}_3$$

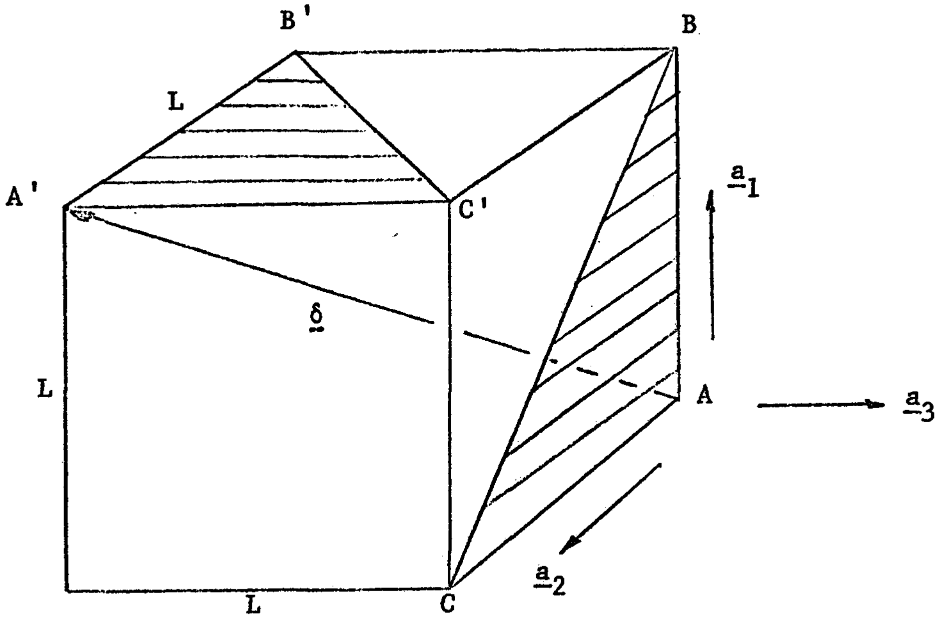
B'
B
L
A'
C'
$\underline{a}_1$
$\underline{\delta}$
L
A
$\underline{a}_3$
$\underline{a}_2$
L
C

Figure 1.9.2

Next, let $\underline{\delta}$ denote the displacement vector of point A (see Fig. 1.9.2); that is let

$$\underline{\delta} = L(\underline{a}_1 + \underline{a}_2 - \underline{a}_3)$$

Then

$$\underline{\rho} \times \underline{\delta} = 2L(\underline{a}_1 + \underline{a}_3)$$

$$(\underline{\rho} \times \underline{\delta}) \times \underline{\rho} = 2L(\underline{a}_1 + 2\underline{a}_2 - \underline{a}_3)$$

and the position vector $\underline{a}^*$ of any point P^* on the screw axis relative to point A prior to the displacement of the triangle is given by

$$\underline{a}^* \underset{(1)}{=} \frac{2L(\underline{a}_1 + \underline{a}_3) + 2L(\underline{a}_1 + 2\underline{a}_2 - \underline{a}_3)}{(2)(3)} + \mu\underline{\rho}$$

$$= \frac{2L}{3}(\underline{a}_1 + \underline{a}_2) + \mu\underline{\rho}$$

Hence, if μ is arbitrarily taken equal to zero, then P^* is situated as shown in Fig. 1.9.3 when the triangle is in its original position, and the screw axis, being parallel to ρ, appears as indicated. Furthermore, the screw translation vector $\underline{\delta}^*$ is given by

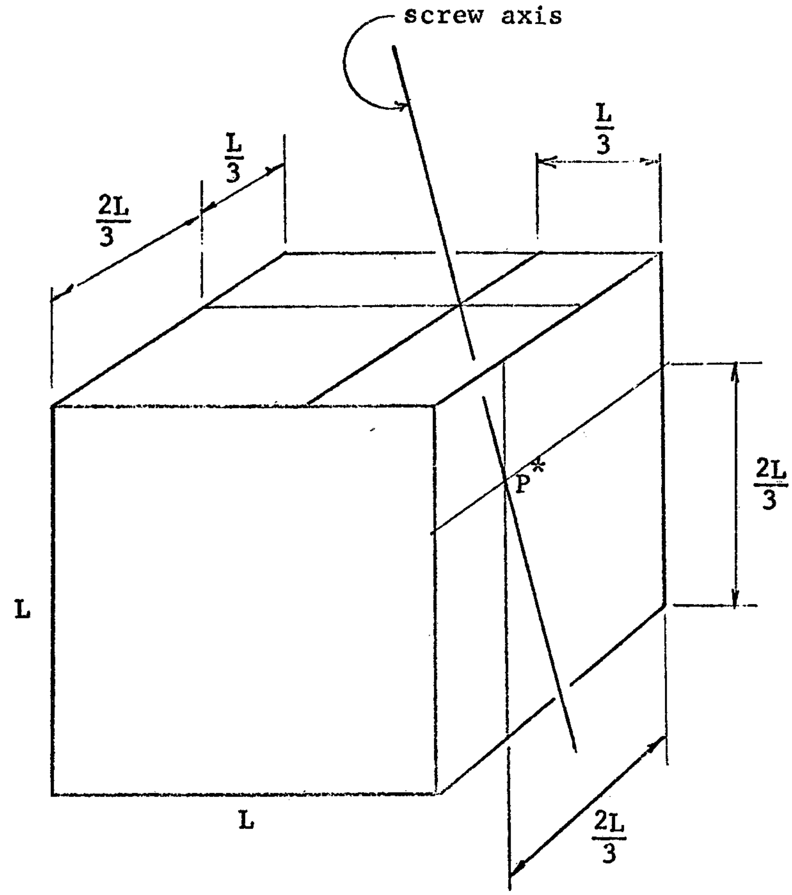

Figure 1.9.3

$$\underline{\delta}^*_{(2)} = \frac{L(\underline{a}_1 - \underline{a}_2 - \underline{a}_3) \cdot (\underline{a}_1 + \underline{a}_2 - \underline{a}_3)}{3} (\underline{a}_1 - \underline{a}_2 - \underline{a}_3)$$

$$= \frac{L}{3} (\underline{a}_1 - \underline{a}_2 - \underline{a}_3)$$

and thus has a magnitude

$$\underline{\delta}^* = \frac{\sqrt{3}}{3} L$$

while the amount θ of the rotation associated with the displacement of the triangle, found in the Example in Sec. 1.3, is given by

$$\theta = \frac{2\pi}{3} \text{ rad}$$

Hence, to bring the triangle into the desired position, one may proceed as follows: Perform a translation through a distance $\sqrt{3}L/3$, as indicated in Fig. 1.9.4, and follow this with a rotation of amount $2\pi/3$ rad. about the screw axis, choosing the sense of the rotation as shown in Fig. 1.9.4.

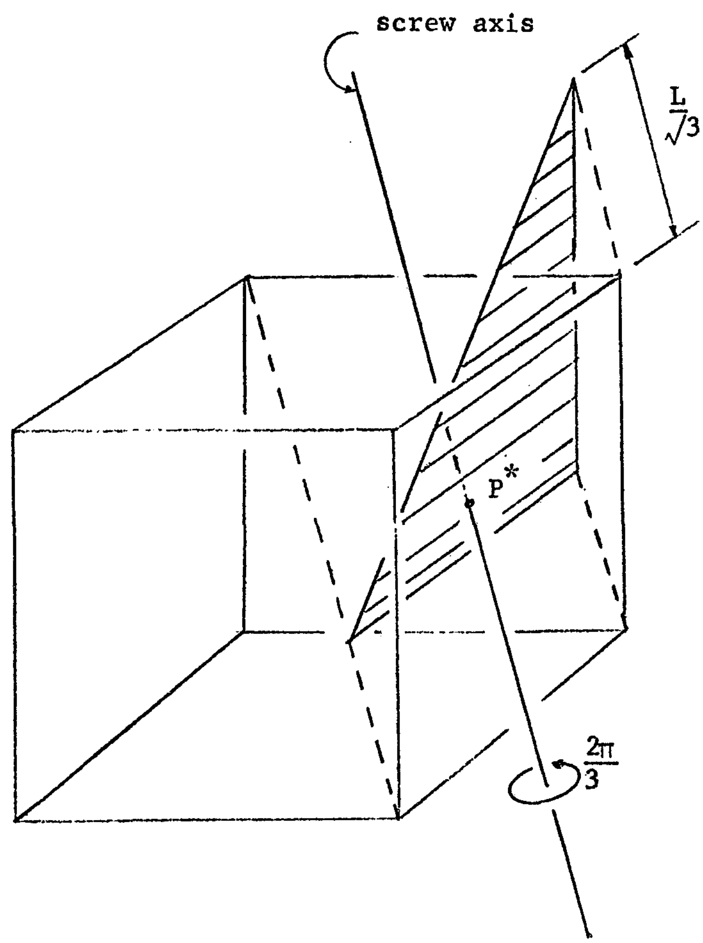

Figure 1.9.4

1.10 Angular velocity matrix

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are two dextral sets of orthogonal unit vectors fixed respectively in two reference frames or rigid bodies A and B which are moving relative to each other, then the direction cosine matrix C and its elements C_{ij} $(i,j = 1,2,3)$, defined in Sec. 1.2, are functions of time t . The time derivative of C , denoted by $\dot{C}$ and defined in terms of the time derivatives $\dot{C}_{ij}$ of C_{ij} $(i,j = 1,2,3)$ as

$$\dot{C} \triangleq \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix}$$

can be expressed as the product of C and a skew-symmetric matrix $\tilde{\omega}$ called an angular velocity matrix for B in A and defined as

$$\tilde{\omega} \triangleq C^T \dot{C} \tag{2}$$

In other words, with $\tilde{\omega}$ defined as in Eq. (2),

$$\dot{C} = C\tilde{\omega} \tag{3}$$

If functions $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are introduced in accordance with the notational convention established in Eq. (1.8.6), that is by expressing $\tilde{\omega}$ as

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4}$$

then ω_1 , ω_2 , and ω_3 are given by

$$\omega_1 = C_{13}\dot{C}_{12} + C_{23}\dot{C}_{22} + C_{33}\dot{C}_{32} \tag{5}$$

$$\omega_2 = C_{21}\dot{C}_{23} + C_{31}\dot{C}_{33} + C_{11}\dot{C}_{13} \tag{6}$$

$$\omega_3 = C_{32}\dot{C}_{31} + C_{12}\dot{C}_{11} + C_{22}\dot{C}_{21} \tag{7}$$

These equations can be expressed more concisely after defining η_{ij} as

$$\eta_{ijk} \triangleq \frac{1}{2}\, \varepsilon_{ijk}\, (\varepsilon_{ijk} + 1) \quad (i,j = 1,2,3) \tag{8}$$

where ε_{ijk} is given by Eq. (1.2.32). (The quantity η_{ijk} is equal to unity when the subscripts appear in cyclic order; otherwise it is equal to zero.) Using the summation convention for repeated subscripts, one can then replace Eqs. (5) - (7) with

$$\omega_i = \eta_{igh}\dot{C}_{jg}C_{jh} \quad (i = 1,2,3) \tag{9}$$

Similarly, Eqs. (3) can be expressed as

$$\dot{C}_{ij} = \varepsilon_{ghi} C_{ig} \omega_h \quad (i,j = 1,2,3) \tag{10}$$

Eqs. (10) are known as Poisson's kinematical equations.

Derivations: Pre-multiplication of $\tilde{\omega}$ with C gives

$$C\,\tilde{\omega} \underset{(2)}{=} C\,C^T\,\dot{C} \underset{(1.2.16)}{=} \dot{C}$$

in agreement with Eq. (3).

To see that $C^T\,\dot{C}$ is skew-symmetric, note that

$$(C^T\,\dot{C})^T + C^T\,\dot{C} = \dot{C}^T\,C + C^T\,\dot{C}$$

$$= \frac{d}{dt}\,(C^T\,C) \underset{(1.2.17)}{=} \frac{dU}{dt} = 0$$

Eqs. (5) - (7) follow from Eqs. (2) and (4), that is, from

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_2 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{32} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix}$$

Example: The quantities ω_1, ω_2, and ω_3 can be expressed in a simple and revealing form when a body B performs a motion of simple rotation (see Sec. 1.1) in a reference frame A. For, letting θ and

λ_i $(i = 1,2,3)$ have the same meaning as in Secs. 1.1 and 1.2, and substituting from Eqs. (1.2.23) - (1.2.31) into Eq. (5), one obtains

$$\begin{aligned}
\omega_1 &= [\lambda_2 \sin\theta + \lambda_3\lambda_1 (1 - \cos\theta)](-\lambda_3 \cos\theta + \lambda_1\lambda_2 \sin\theta)\dot{\theta} \\
&\quad - [-\lambda_1 \sin\theta + \lambda_2\lambda_3 (1 - \cos\theta)](\lambda_3^{\,2} + \lambda_1^{\,2}) \sin\theta\, \dot{\theta} \\
&\quad + [1 - (\lambda_1^{\,2} + \lambda_2^{\,2})(1 - \cos\theta)](\lambda_1 \cos\theta + \lambda_2\lambda_3 \sin\theta)\dot{\theta} \\
&= [\lambda_1 (\lambda_1^{\,2} + \lambda_2^{\,2} + \lambda_3^{\,2}) \\
&\quad + (1 - \lambda_1^{\,2} - \lambda_2^{\,2} - \lambda_3^{\,2})(\lambda_1 \cos\theta + \lambda_2\lambda_3 \sin\theta - \lambda_2\lambda_3 \sin\theta\cos\theta)]\dot{\theta}
\end{aligned}$$

which, since

$$\lambda_1^{\,2} + \lambda_2^{\,2} + \lambda_3^{\,2} = 1$$

reduces to

$$\omega_1 = \lambda_1\dot{\theta} \tag{a}$$

Similarly,

$$\omega_2 = \lambda_2\dot{\theta} \tag{b}$$

and

$$\omega_3 = \lambda_3\dot{\theta} \tag{c}$$

1.11 Angular velocity vector

The vector $\underline{\omega}$ defined as

$$\underline{\omega} \overset{\Delta}{=} \omega_1\underline{b}_1 + \omega_2\underline{b}_2 + \omega_3\underline{b}_3 \tag{1}$$

where ω_i and $\underline{b}_i$ $(i = 1,2,3)$ have the same meaning as in Sec. 1.10, is called the angular velocity of B in (or relative to) A . At times it is convenient to use the more elaborate symbol ${}^A\underline{\omega}^B$ in place of $\underline{\omega}$. The symbol ${}^B\underline{\omega}^A$ then denotes the angular velocity of A in B , and

$${}^A\underline{\omega}^B = -{}^B\underline{\omega}^A \tag{2}$$

If the first time-derivative of $\underline{b}_i$ in reference frame A is denoted by $\dot{\underline{b}}_i$, that is, $\dot{\underline{b}}_i$ is defined as

$$\dot{\underline{b}}_i = \underline{a}_j \frac{d}{dt}(\underline{a}_j \cdot \underline{b}_i) \quad (i = 1,2,3) \tag{3}$$

where the summation convention for repeated subscripts is used and $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ form a dextral set of orthogonal unit vectors fixed in A , then $\underline{\omega}$ can be expressed as

$$\underline{\omega} = \underline{b}_1\, \dot{\underline{b}}_2 \cdot \underline{b}_3 + \underline{b}_2\, \dot{\underline{b}}_3 \cdot \underline{b}_1 + \underline{b}_3\, \dot{\underline{b}}_1 \cdot \underline{b}_2 \tag{4}$$

When the motion of B in A is one of simple rotation (see Sec.

1.1), the angular velocity of B in A becomes

$$\underline{\omega} = \dot{\theta}\,\underline{\lambda} \tag{5}$$

where θ and λ have the same meaning as in Sec. 1.1.

One of the most useful relationships involving angular velocity is that between the first time-derivatives of a vector $\underline{v}$ in two reference frames A and B . If these derivatives are denoted by $\frac{{}^A d\underline{v}}{dt}$ and $\frac{{}^B d\underline{v}}{dt}$, that is,

$$\frac{{}^A d\underline{v}}{dt} \overset{\Delta}{=} \underline{a}_i \frac{d}{dt}(\underline{v} \cdot \underline{a}_i) \tag{6}$$

and

$$\frac{{}^B d\underline{v}}{dt} \overset{\Delta}{=} \underline{b}_i \frac{d}{dt}(\underline{v} \cdot \underline{b}_i) \tag{7}$$

then this relationship assumes the form

$$\frac{{}^A d\underline{v}}{dt} = \frac{{}^B d\underline{v}}{dt} + {}^A\underline{\omega}^B \times \underline{v} \tag{8}$$

Applied to a vector $\underline{\beta}$ fixed in B , Eq. (8) gives

$$\frac{{}^{A}d\underline{\beta}}{dt} = {}^{A}\underline{\omega}^{B} \times \underline{\beta} \qquad (9)$$

In view of this result one may regard the angular velocity of B in A as an "operator" which, when operating on any vector fixed in B , produces the time-derivative of that vector in A .

Derivatives: For $i = 2$, the scalar product apearing in Eq. (3) can be expressed as

$$\underline{a}_j \cdot \underline{b}_2 \underset{(1.2.1)}{=} C_{j2}$$

Consequently,

$$\dot{\underline{b}}_2 \underset{(3)}{=} \underline{a}_1\dot{C}_{12} + \underline{a}_2\dot{C}_{22} + \underline{a}_3\dot{C}_{32}$$

and, expressing $\underline{b}_3$ as

$$\underline{b}_3 \underset{(1.2.1)}{=} \underline{a}_1 C_{13} + \underline{a}_2 C_{23} + \underline{a}_3 C_{33}$$

one finds that

$$\dot{\underline{b}}_2 \cdot \underline{b}_3 = C_{13}\dot{C}_{12} + C_{23}\dot{C}_{22} + C_{33}\dot{C}_{32} \underset{(1.10.5)}{=} \omega_1$$

Similarly,

$$\dot{\underline{b}}_3 \cdot \underline{b}_1 = \omega_2$$

and

$$\dot{\underline{b}}_1 \cdot \underline{b}_2 = \omega_3$$

Substituting into Eq. (1), one thus arrives at Eq. (4).

When the motion of B in A is one of simple rotation, Eqs. (a) - (c) of the Example in Sec. 1.10 may be used to express the angular velocity of B in A as

$$\underline{\omega} \underset{(1)}{=} (\lambda_1\underline{b}_1 + \lambda_2\underline{b}_2 + \lambda_3\underline{b}_3)\,\dot{\theta}$$

$$\underset{(1.2.22)}{=} (\underline{\lambda} \cdot \underline{b}_1\underline{b}_1 + \underline{\lambda} \cdot \underline{b}_2\underline{b}_2 + \underline{\lambda} \cdot \underline{b}_3\underline{b}_3)\,\dot{\theta} = \underline{\lambda}\dot{\theta}$$

in agreement with Eq. (5) .

To establish the validity of Eq. (8), let ${}^{A}v_i$, ${}^{B}v_i$, ${}^{A}v$, and ${}^{B}v$ have the same meaning as in Sec. 1.2. Then, from Eqs. (6) and (1.2.7),

$$\frac{{}^{A}d\underline{v}}{dt} = {}^{A}\dot{v}_1\underline{a}_1 + {}^{A}\dot{v}_2\underline{a}_2 + {}^{A}\dot{v}_3\underline{a}_3$$

$$= {}^{A}\dot{v}[\underline{a}_1\ \underline{a}_2\ \underline{a}_3]^T$$

Now,

$$ {}^{A}\dot{v} \underset{(1.2.9)}{=} \frac{d}{dt}\left({}^{B}vC^{T}\right) = {}^{B}\dot{v}C^{T} + {}^{B}v\dot{C}^{T} $$

and

$$ [\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3]^T \underset{(1.2.2)}{=} C\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T $$

Hence

$$ \frac{{}^{A}d\underline{v}}{dt} = {}^{B}\dot{v}C^{T}\,C\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T + {}^{B}v\dot{C}^{T}\,C\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T $$

$$ \underset{(1.2.17,\ 1.10.2)}{=} {}^{B}\dot{v}[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T + {}^{B}v\,\tilde{\omega}^T\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T $$

Furthermore, from Eqs. (7) and (1.2.8)

$$ {}^{B}\dot{v}\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T = \frac{{}^{B}d\underline{v}}{dt} $$

while it follows from Eqs. (1) and (1.10.4) that

$$ {}^{B}v\tilde{\omega}^T\,[\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3]^T = {}^{A}\underline{\omega}^{B} \times \underline{v} $$

Consequently

$$ \frac{{}^{A}d\underline{v}}{dt} = \frac{{}^{B}d\underline{v}}{dt} + {}^{A}\underline{\omega}^{B} \times \underline{v} $$

Finally, Eq. (2) follows from the fact that, interchanging A and B in Eq. (8), one obtains

$$\frac{{}^{B}d\underline{v}}{dt} = \frac{{}^{A}d\underline{v}}{dt} + {}^{B}\underline{\omega}^{A} \times \underline{v}$$

and, adding corresponding members of this equation and of Eq. (8), one arrives at

$$\frac{{}^{A}d\underline{v}}{dt} + \frac{{}^{B}d\underline{v}}{dt} = \frac{{}^{B}d\underline{v}}{dt} + \frac{{}^{A}d\underline{v}}{dt} + ({}^{A}\underline{\omega}^{B} + {}^{B}\underline{\omega}^{A}) \times \underline{v}$$

or

$$({}^{A}\underline{\omega}^{B} + {}^{B}\underline{\omega}^{A}) \times \underline{v} = 0$$

This equation can be satisfied for *all* $\underline{v}$ only if

$${}^{A}\underline{\omega}^{B} = -{}^{B}\underline{\omega}^{A}$$

Example: When a point P moves on a space curve C fixed in a reference frame A (see Fig. 1.11.1), a dextral set of orthogonal unit vectors $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ can be generated by letting $\underline{p}$ be the position vector of P relative to a point P_o fixed on C and defining $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ as

$$\underline{b}_1 \triangleq \underline{p}' \tag{a}$$

$$\underline{b}_2 \triangleq \underline{p}''/|\underline{p}'| \tag{b}$$

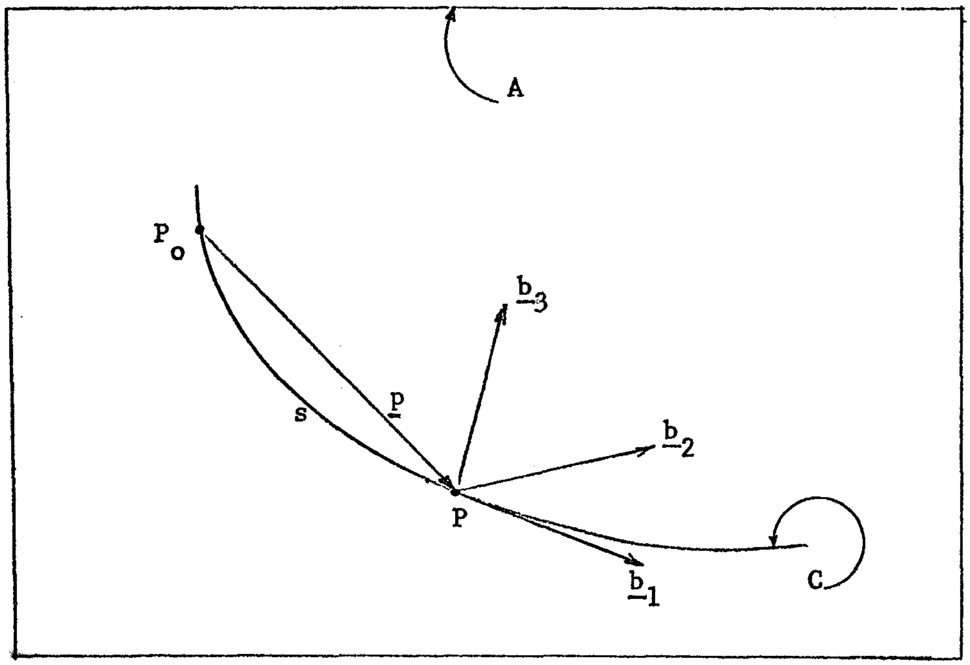

Figure 1.11.1

$$\underline{b}_3 \triangleq \underline{p}' \times \underline{p}''/|\underline{p}''| \tag{c}$$

where primes denote differentation in A with respect to the arc length displacement s of P relative to P_o . The vector $\underline{b}_1$ is called a vector tangent, $\underline{b}_2$ the vector principal normal, and $\underline{b}_3$ a vector binormal of C at P ; and the derivatives of $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ with respect to s are given by the Serret-Frênet formulas

$$\underline{b}_1' = \underline{b}_2/\rho \tag{d}$$

$$\underline{b}_2' = -\underline{b}_1/\rho + 2\underline{b}_3 \tag{e}$$

$$\underline{b}_3' = -\lambda\underline{b}_2 \tag{f}$$

where ρ and λ , defined as

$$\rho \triangleq 1/1\underline{p}'' \tag{g}$$

and

$$\lambda \triangleq \rho^2\underline{p}' \cdot \underline{p}'' \times \underline{p}''' \tag{h}$$

are called the principal radius of curvature of C at P and the torsion of C at P .

If B designates a reference frame in which $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are fixed, the angular velocity $\underline{\omega}$ of B in A can be expressed in terms of $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, ρ , λ , and $\dot{s}$ by using Eq. (4) together with

$$\dot{\underline{b}}_1 = \underline{b}_1{}'\dot{s} \underset{(d)}{=} \underline{b}_2\dot{s}/\rho \tag{i}$$

$$\dot{\underline{b}}_2 \underset{(e)}{=} (-\underline{b}_1/\rho + \lambda\underline{b}_3)\dot{s} \tag{j}$$

$$\dot{\underline{b}}_3 \underset{(f)}{=} -\lambda\underline{b}_2\dot{s} \tag{k}$$

$$\dot{\underline{b}}_1 \cdot \underline{b}_2 \underset{(i)}{=} \dot{s}/\rho \ , \qquad \dot{\underline{b}}_2 \cdot \underline{b}_3 \underset{(j)}{=} \lambda\dot{s} \ , \qquad \dot{\underline{b}}_3 \cdot \underline{b}_1 \underset{(k)}{=} 0$$

to obtain

$$\underline{\omega} \underset{(4)}{=} (\lambda\underline{b}_1 + \underline{b}_3/\rho)\dot{s}$$

The term "torsion" as applied to λ is seen to be particularly appropriate in this context.

1.12 Angular velocity components

The expression for $\underline{\omega}$ given in Eq. (1.11.1) involves three components, each of which is parallel to a unit vector fixed in B . At times it is necessary to express $\underline{\omega}$ in other ways, for example, to resolve it into components parallel to unit vectors fixed in A . Whichever resolution is employed, one may wish to know what the physical significance of any one component of $\underline{\omega}$ is.

In certain situations physical significance can be attributed to angular velocity components by identifying for each component two reference frames such that the angular velocity of one of these relative to the other is equal to the component in question. As will be seen later, this is the case, for example, when the angular velocity of B in A is expressed as in Eq. (1.16.1). In general, however, it is not a simple matter to discover the necessary reference frames. For instance, such reference frames are not readily identifiable for the components $\lambda\dot{s}\underline{b}_1$ and $(\dot{s}/\rho)\underline{b}_3$ of the angular velocity found in the Example in Sec. 1.11.

An essentially geometric interpretation can be given to the quantities ω_1 , ω_2 , and ω_3 appearing in Eq. (1.11.1), and thus to the components $\omega_i\underline{b}_i$ $(i = 1,2,3)$ of $\underline{\omega}$, by introducing a certain space-average value of the first time-derivative of each of three angles*. Specifically, let $\underline{\alpha}$ be a generic unit vector fixed in reference frame A , $\underline{\beta}_i$ the orthogonal projection of $\underline{\alpha}$ on a plane normal to $\underline{b}_i$ $(i = 1,2,3)$, θ_1 the angle between $\underline{\beta}_1$ and $\underline{b}_3$, θ_2 the angle

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*The authors are indebted to Professor R. Skalak of Columbia University for this idea.

between β_2 and $\underline{b}_1$, and θ_3 the angle between β_3 and $\underline{b}_2$, as shown in Fig. 1.12.1. Next, letting S be a unit sphere centered at a point O , and designating as P the point of S whose position vector relative to O is parallel to $\underline{\alpha}$ (see Fig. 1.12.2), associate with P the value of $\dot{\theta}_i$ and define $\bar{\dot{\theta}}_i$ as

$$\bar{\dot{\theta}}_i \triangleq \frac{1}{4\pi} \int \dot{\theta}_i \, d\sigma \qquad (i = 1,2,3) \tag{1}$$

where $d\sigma$ in the area of a differential element of S at P . Then

$$\omega_i = \bar{\dot{\theta}}_i \qquad (i = 1,2,3) \tag{2}$$

<u>Derivation</u>: Defining α_i as

$$\alpha_i \triangleq \underline{\alpha} \cdot \underline{b}_i \qquad (i = 1,2,3)$$

one can express θ_1 (see Fig. 1.12.1) as

$$\theta_1 = \text{arc tan} \ (\alpha_2/\alpha_3)$$

from which it follows that

$$\dot{\theta}_1 = \frac{\dot{\alpha}_2\alpha_3 - \dot{\alpha}_3\alpha_2}{\alpha_2{}^2 + \alpha_3{}^2}$$

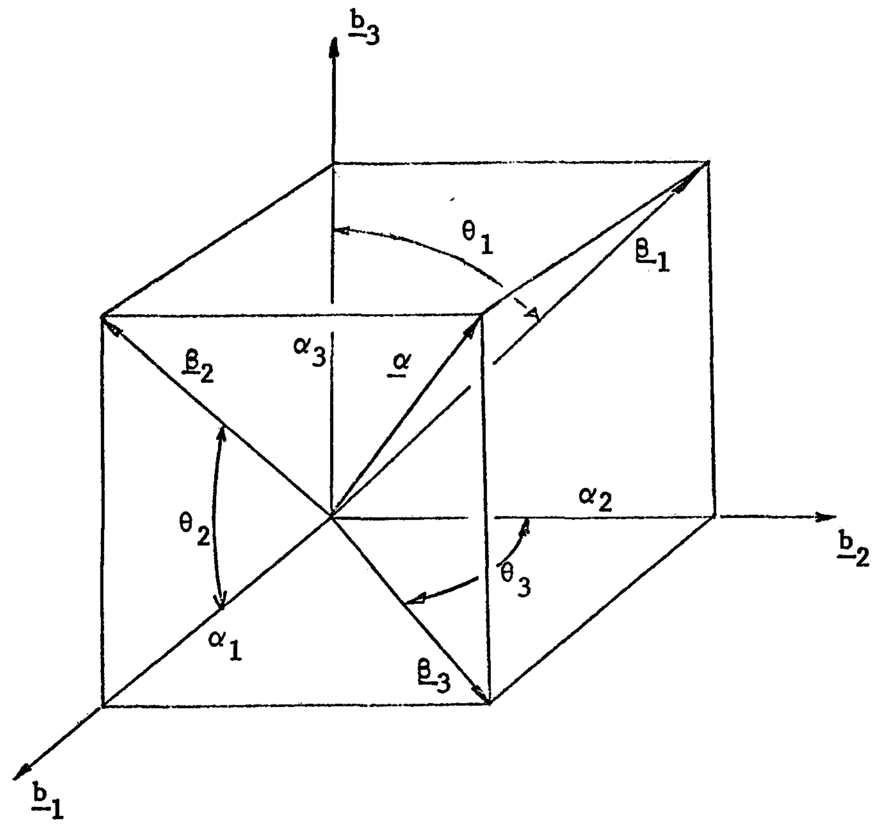

$\underline{b}_3$
θ_1
$\underline{\beta}_1$
α_3
$\underline{\alpha}$
$\underline{\beta}_2$
θ_2
α_2
$\underline{b}_2$
θ_3
α_1
$\underline{\beta}_3$
$\underline{b}_1$

Figure 1.12.1

Now, $\frac{{}^B d\underline{\underline{\alpha}}}{dt}$ is given both by

$$\frac{{}^B d\underline{\underline{\alpha}}}{dt} = \dot{\alpha}_1 \underline{b}_1 + \dot{\alpha}_2 \underline{b}_2 + \dot{\alpha}_3 \underline{b}_3$$

and by

$$\frac{{}^B d\underline{\underline{\alpha}}}{dt} \underset{(1.11.9)}{=} {}^B\underline{\omega}^A \times \underline{\alpha} \underset{(1.11.2)}{=} - {}^A\underline{\omega}^B \times \underline{\alpha}$$

$$\underset{(1.11.1)}{=} (\alpha_2 \omega_3 - \alpha_3 \omega_2) \underline{b}_1$$

$$+ (\alpha_3 \omega_1 - \alpha_1 \omega_3) \underline{b}_2$$

$$+ (\alpha_1 \omega_2 - \alpha_2 \omega_1) \underline{b}_3$$

Consequently

$$\dot{\alpha}_i \underset{(1.10.8)}{=} \eta_{ijk} (\alpha_j \omega_k - \alpha_k \omega_j) \quad (i = 1,2,3)$$

and

$$\dot{\theta}_1 = \frac{(\alpha_3 \omega_1 - \alpha_1 \omega_3) \alpha_3 - (\alpha_1 \omega_2 - \alpha_2 \omega_1) \alpha_2}{\alpha_2^2 + \alpha_3^2}$$

$$= \omega_1 - \frac{\alpha_1 \alpha_2}{\alpha_2^2 + \alpha_3^2} \omega_2 - \frac{\alpha_1 \alpha_3}{\alpha_2^2 + \alpha_3^2} \omega_3$$

To perform the integration indicated in Eq. (1), introduce the angles ϕ and ψ shown in Fig. 1.12.2, noting that $\underline{\alpha}$ then can be expressed as

$$\underline{\alpha} = \cos\phi\ \underline{b}_1 + \sin\phi\cos\psi\ \underline{b}_2 + \sin\phi\sin\psi\ \underline{b}_3$$

so that

$$\alpha_1 = \cos\phi\ , \ \alpha_2 = \sin\phi\cos\psi\ , \ \alpha_3 = \sin\phi\sin\psi$$

while

$$d\sigma = \sin\phi\ d\phi\ d\psi$$

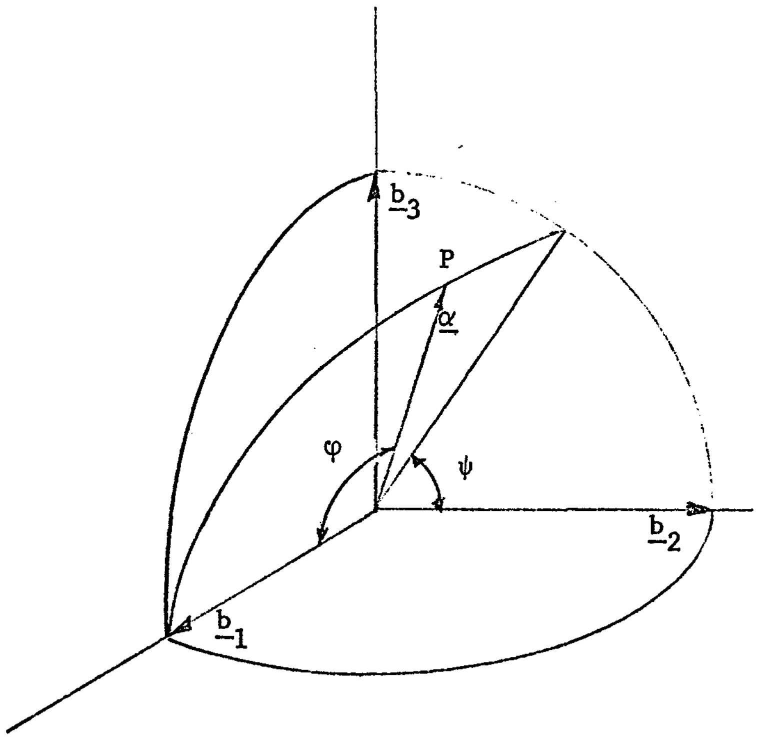
Consequently

$$4\pi\,\overline{\dot{\theta}}_1 \underset{(1)}{=} \omega_1 \int_0^{\pi}\left[\int_0^{2\pi} \sin\phi\ d\psi\right] d\phi - \omega_2 \int_0^{\pi}\left[\int_0^{2\pi} \cos\phi\cos\psi\ d\psi\right] d\phi$$

$$- \omega_3 \int_0^{\pi}\left[\int_0^{2\pi} \cos\phi\sin\psi\ d\psi\right] d\phi$$

The first integral has the value 4π, and the remaining two integrals are equal to zero. Hence

$$\overline{\dot{\theta}}_1 = \omega_1$$



$\underline{b}_3$
P
$\underline{\alpha}$
φ
ψ
$\underline{b}_2$
$\underline{b}_1$

Figure 1.12.2

Similarly,

$$\bar{\dot{\theta}}_2 = \omega_2$$

and

$$\bar{\dot{\theta}}_3 = \omega_3$$

<u>Example</u>: In Fig. 1.12.3, B designates a cylindrical spacecraft whose attitude motion in a reference frame A can be described as a combination of "coning" and "spinning", the former being characterized by the angle ϕ and involving the motion of the symmetry axis of B on the surface of a cone that is fixed in A and has a constant semi-vertex angle θ, while the latter is associated with changes in the angle ψ between two lines which intersect on, and are perpendicular to, the symmetry axis of B, one line being fixed in B and the other one intersecting the axis of the cone. Under these circumstances the direction cosine matrix C such that

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]\ C \tag{a}$$

where $\underline{a}_i$ and $\underline{b}_i$ $(i = 1,2,3)$ are unit vectors directed as shown in Fig. 1.12.3, can be expressed as

$$C \underset{(1.6.15)}{=} C_1\ (\phi)\ C_3\ (\theta)\ C_1\ (\psi)$$

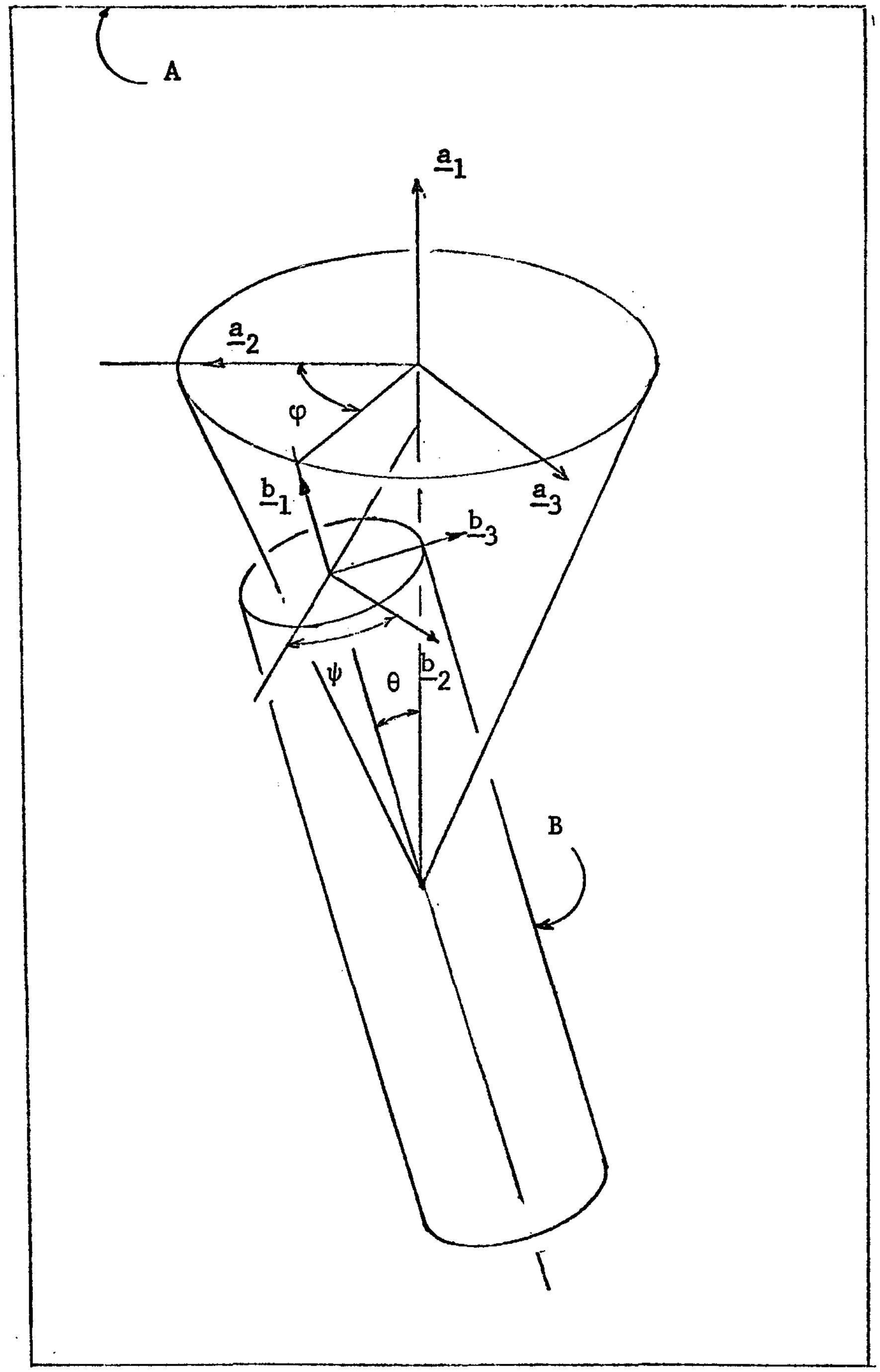

A
$\underline{a}_1$
$\underline{a}_2$
φ
$\underline{b}_1$
$\underline{a}_3$
$\underline{b}_3$
ψ
θ
$\underline{b}_2$
B

Figure 1.12.3

or, after using Eqs. (1.2.35) and (1.2.37), as

$$C = \begin{bmatrix} c\theta & -s\theta c\psi & s\theta s\psi \\ s\theta c\phi & c\theta c\phi c\psi - s\phi s\psi & -c\theta c\phi s\psi - s\phi c\psi \\ s\theta s\phi & c\theta s\phi c\psi + c\phi s\psi & -c\theta s\phi s\psi + c\phi c\psi \end{bmatrix} \tag{b}$$

where $s\theta$ and $c\theta$ denote $\sin\theta$ and $\cos\theta$, respectively, and similarly for ϕ and ψ. Forming ω_1, ω_2, and ω_3 in accordance with Eqs. (1.10.5) - (1.10.7), one then obtains the following expression for the angular velocity $\underline{\omega}$ of B in A :

$$\underline{\omega} = (\dot{\psi} + \dot{\phi}c\theta)\,\underline{b}_1 - \dot{\phi}s\theta c\psi\,\underline{b}_2 + \dot{\phi}s\theta s\psi\,\underline{b}_3 \tag{c}$$

A more efficient method for obtaining this result is described in Sec. 1.16. For present purposes, what is of interest is the fact that the $\underline{b}_i$ -component ($i = 1,2,3$) in Eq. (c) does not have a readily apparent physical significance, but that, when $\underline{\omega}$ is re-written as

$$\underline{\omega} = \dot{\psi}\,\underline{b}_1 + \dot{\phi}(c\theta\ b_1 + s\theta c\phi\,\underline{b}_2 + s\theta s\phi\,\underline{b}_3)$$

$$\underset{(a,b)}{=} \dot{\psi}\,\underline{b}_1 + \dot{\phi}\,\underline{a}_1$$

then each component has the same form as the right-hand member of Eq. (1.11.5) and can, therefore, be regarded as the angular velocity of a body performing a motion of simple rotation. Specifically, designating as A_1 a reference frame in which the axis of the cone and the symmetry axis of the cone and the symmetry axis of B are fixed, one can observe that A_1 performs a motion of simple rotation in A ; moreover, that B performs such a motion in A_1 ; and, finally, that the associated angular velocities are

$$ {}^{A}\underline{\omega}^{A_1} = \dot{\phi}\,\underline{a}_1 $$

and

$$ {}^{A_1}\underline{\omega}^{B} = \dot{\psi}\,\underline{b}_1 $$

Thus it appears that

$$ \underline{\omega} = {}^{A}\underline{\omega}^{A_1} + {}^{A_1}\underline{\omega}^{B} $$

1.13 Angular velocity and Euler parameters

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are two dextral sets of orthogonal unit vectors fixed respectively in reference frames or rigid bodies A and B which are moving relative to each other, one can use Eqs. (1.3.15) - (1.3.18) to associate with each instant of time Euler parameters $\varepsilon_1,\ldots,\varepsilon_4$; and an Euler vector $\underline{\varepsilon}$ can then be formed by reference to Eq. (1.3.2). In terms of $\underline{\varepsilon}$ and ε_4 , the angular velocity of B in A (see Sec. 1.11) can be expressed as

$$\underline{\omega} = 2\left(\varepsilon_4 \frac{{}^{B}d\underline{\varepsilon}}{dt} - \dot{\varepsilon}_4\underline{\varepsilon} - \underline{\varepsilon} \times \frac{{}^{B}d\underline{\varepsilon}}{dt}\right) \tag{1}$$

Conversely, if $\underline{\omega}$ is known as a function of time, the Euler parameters can be found by solving the differential equations

$$\frac{{}^{B}d\underline{\varepsilon}}{dt} = \frac{1}{2}\,(\varepsilon_4\underline{\omega} + \underline{\varepsilon} \times \underline{\omega}) \tag{2}$$

and

$$\dot{\varepsilon}_4 = -\frac{1}{2}\,\underline{\omega} \cdot \underline{\varepsilon} \tag{3}$$

Equations equivalent to Eqs. (1) - (3) can be formulated in terms of matrices ω , ε , and E defined as

$$\omega \triangleq [\omega_1 \quad \omega_2 \quad \omega_3 \quad 0] \tag{4}$$

$$\varepsilon \triangleq [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4] \tag{5}$$

and

$$E = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix} \qquad (6)$$

These equations are

$$\omega = 2\dot{\varepsilon}E \qquad (7)$$

and

$$\dot{\varepsilon} = \frac{1}{2}\omega E^T \qquad (8)$$

<u>Derivations</u>: Substitution from Eqs. (1.3.6) - (1.3.14) into Eqs. (1.10.6) - (1.10.8) gives

$$\begin{aligned} \omega_1 &= 4(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)(\dot{\varepsilon}_1\varepsilon_2 + \varepsilon_1\dot{\varepsilon}_2 - \dot{\varepsilon}_3\varepsilon_4 - \varepsilon_3\dot{\varepsilon}_4) \\ &\quad + 4(\varepsilon_2\varepsilon_2 - \varepsilon_1\varepsilon_4)(\varepsilon_2\dot{\varepsilon}_2 - \varepsilon_3\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_1 + \varepsilon_4\dot{\varepsilon}_4) \\ &\quad + 2(1 - 2\varepsilon_1^{\,2} - 2\varepsilon_2^{\,2})(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4) \\ &= 2(\dot{\varepsilon}_1\varepsilon_4 + \dot{\varepsilon}_2\varepsilon_3 - \dot{\varepsilon}_3\varepsilon_2 - \dot{\varepsilon}_4\varepsilon_1) \end{aligned}$$

$$\omega_2 = 2(\dot{\varepsilon}_2\varepsilon_4 + \dot{\varepsilon}_3\varepsilon_1 - \dot{\varepsilon}_1\varepsilon_3 - \dot{\varepsilon}_4\varepsilon_2)$$

$$\omega_3 = 2(\dot{\varepsilon}_3\varepsilon_4 + \dot{\varepsilon}_1\varepsilon_2 - \dot{\varepsilon}_2\varepsilon_1 - \dot{\varepsilon}_4\varepsilon_3)$$

and these are three of the four scalar equations corresponding to Eq. (7). The fourth is

$$0 \underset{(7)}{=} 2(\dot{\varepsilon}_1\varepsilon_1 + \dot{\varepsilon}_2\varepsilon_2 + \dot{\varepsilon}_3\varepsilon_3 + \dot{\varepsilon}_4\varepsilon_4)$$

and this equation is satisfied because

$$\dot{\varepsilon}_1\varepsilon_1 + \dot{\varepsilon}_2\varepsilon_2 + \dot{\varepsilon}_3\varepsilon_3 = \frac{1}{2}\frac{d}{dt}(\varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2 + \varepsilon_4{}^2) \underset{(1.3.4)}{=} 0$$

Thus the validity of Eq. (7) is established; and Eq. (1) can be obtained by noting that

$$\begin{aligned}
\underline{\omega} &\underset{(1.11.1)}{=} \omega_1\underline{b}_1 + \omega_2\underline{b}_2 + \omega_3\underline{b}_3 \\
&\underset{(7)}{=} 2[(\dot{\varepsilon}_1\varepsilon_4 + \dot{\varepsilon}_2\varepsilon_3 - \dot{\varepsilon}_3\varepsilon_2 - \dot{\varepsilon}_4\varepsilon_1)\underline{b}_1 \\
&\quad + (\dot{\varepsilon}_2\varepsilon_4 + \dot{\varepsilon}_3\varepsilon_1 - \dot{\varepsilon}_1\varepsilon_3 - \dot{\varepsilon}_4\varepsilon_2)\underline{b}_2 \\
&\quad + (\dot{\varepsilon}_3\varepsilon_4 + \dot{\varepsilon}_1\varepsilon_2 - \dot{\varepsilon}_2\varepsilon_1 - \dot{\varepsilon}_4\varepsilon_3)\underline{b}_3] \\
&= 2[\varepsilon_4(\dot{\varepsilon}_1\underline{b}_1 + \dot{\varepsilon}_2\,\underline{b}_2 + \dot{\varepsilon}_3\underline{b}_3) - \dot{\varepsilon}_4(\varepsilon_1\underline{b}_1 + \varepsilon_2\underline{b}_2 + \varepsilon_3\underline{b}_3) \\
&\quad + (\dot{\varepsilon}_2\varepsilon_3 - \dot{\varepsilon}_3\varepsilon_2)\underline{b}_1 + (\dot{\varepsilon}_3\varepsilon_1 - \dot{\varepsilon}_1\varepsilon_3)\underline{b}_2 + (\dot{\varepsilon}_1\varepsilon_2 - \dot{\varepsilon}_2\varepsilon_1)\underline{b}_3] \\
&\underset{(1.3.2)}{=} 2\left(\varepsilon_4\,\frac{{}^B d\underline{\varepsilon}}{dt} - \dot{\varepsilon}_4\underline{\varepsilon} - \underline{\varepsilon} \times \frac{{}^B d\underline{\varepsilon}}{dt}\right)
\end{aligned}$$

Post-multiplication of both sides of Eq. (7) with E^T gives

$$\omega E^T = 2\dot{\varepsilon}EE^T$$

Now, using Eq. (1.3.4) and referring to Eq. (6), one finds that

$$E\,E^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consequently

$$\omega E^T = 2\dot{\varepsilon}$$

in agreement with Eq. (8).

Finally,

$$\dot{\varepsilon}_4 \underset{(8)}{=} -\frac{1}{2}(\omega_1\varepsilon_1 + \omega_2\varepsilon_2 + \omega_3\varepsilon_3) \underset{(1.11.1,1.3.2)}{=} -\frac{1}{2}\underline{\omega} \cdot \underline{\varepsilon}$$

as in Eq. (3); and

$$\begin{aligned} \frac{{}^B d\underline{\varepsilon}}{dt} &\underset{(1.3.2)}{=} \dot{\varepsilon}_1\underline{b}_1 + \dot{\varepsilon}_2\underline{b}_2 + \dot{\varepsilon}_3\underline{b}_3 \\ &\underset{(8)}{=} \frac{1}{2}[(\omega_1\varepsilon_4 - \omega_2\varepsilon_3 + \omega_3\varepsilon_2)\underline{b}_1 \\ &\quad + (\omega_1\varepsilon_3 + \omega_2\varepsilon_4 - \omega_3\varepsilon_1)\underline{b}_2 \\ &\quad - (\omega_1\varepsilon_2 - \omega_2\varepsilon_1 - \omega_3\varepsilon_4)\underline{b}_3] \end{aligned}$$

The right-hand member of this equation is equal to that of Eq. (2).

Example: Suppose that the inertia ellipsoid of B for the mass center B^* of B is an ellipsoid of revolution whose axis of revolution is parallel to $\underline{b}_3$. Then, if $\underline{I}$ denotes the inertia dyadic of B for B^*, and if I and J are defined as

$$I \triangleq \underline{b}_1 \cdot \underline{I} \cdot \underline{b}_1 = \underline{b}_2 \cdot \underline{I} \cdot \underline{b}_2 \qquad \text{(a)}$$

and

$$J \triangleq \underline{b}_3 \cdot \underline{I} \cdot \underline{b}_3 \qquad \text{(b)}$$

the angular momentum $\underline{H}$ of B in A with respect to B^* is given by

$$\underline{H} = I\omega_1\underline{b}_1 + I\omega_2\underline{b}_2 + J\omega_3\underline{b}_3 \qquad \text{(c)}$$

and the first time-derivative of $\underline{H}$ in A can be expressed as

$$\frac{{}^A d\underline{H}}{dt} \underset{(1.11.8)}{=} \frac{{}^B d\underline{H}}{dt} + \underline{\omega} \times \underline{H}$$

$$\underset{(c)}{=} [I\dot{\omega}_1 + (J - I)\omega_2\omega_3]\underline{b}_1 + [I\dot{\omega}_2 - (J - I)\omega_3\omega_1]\underline{b}_2 + J\dot{\omega}_3\underline{b}_3 \qquad \text{(d)}$$

Hence, if B moves under the action of forces the sum of whose moments about B^* is equal to zero, and if A is an inertial reference frame, so that, in accordance with the angular momentum principle, $\frac{{}^A d\underline{H}}{dt}$ is equal to zero, then ω_1, ω_2, and ω_3 are governed by the differential equations

$$\dot{\omega}_1 - \frac{I - J}{I}\omega_2\omega_3 = 0 \qquad \text{(e)}$$

$$\dot{\omega}_2 + \frac{I - J}{I}\omega_3\omega_1 = 0 \qquad \text{(f)}$$

$$\dot{\omega}_3 = 0 \tag{g}$$

Letting $\overline{\omega}_i$ denote the value of ω_i $(i = 1,2,3)$ at $t = 0$, and defining a constant s as

$$s \triangleq \frac{I - J}{I} \overline{\omega}_3 \tag{h}$$

one can express the general solution of Eqs. (e) - (g) as

$$\omega_1 = \overline{\omega}_1 \cos st + \overline{\omega}_2 \sin st \tag{i}$$

$$\omega_2 = -\overline{\omega}_1 \sin st + \overline{\omega}_2 \cos st \tag{j}$$

$$\omega_3 = \overline{\omega}_3 \tag{k}$$

and, to determine the orientation of B in A , one then can seek the solution of the differential equations

$$\dot{\varepsilon}_1 \underset{(8)}{=} \frac{1}{2} (\omega_1\varepsilon_4 - \omega_2\varepsilon_3 + \omega_3\varepsilon_2)$$

$$\underset{(i-k)}{=} \frac{1}{2} [(\overline{\omega}_1 \cos st + \overline{\omega}_2 \sin st)\varepsilon_4 + (\overline{\omega}_1 \sin st - \overline{\omega}_2 \cos st)\varepsilon_3 + \overline{\omega}_3\varepsilon_2] \tag{ℓ}$$

$$\dot{\varepsilon}_2 \underset{(8)}{=} \frac{1}{2} (\omega_1\varepsilon_3 + \omega_2\varepsilon_4 - \omega_3\varepsilon_1)$$

$$\underset{(i-k)}{=} \frac{1}{2} [(\overline{\omega}_1 \cos st + \overline{\omega}_2 \sin st)\varepsilon_3 - (\overline{\omega}_1 \sin st - \overline{\omega}_2 \cos t)\varepsilon_4 - \overline{\omega}_3\varepsilon_1] \tag{m}$$

plus two more differential equations of the same form, using as initial conditions

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0\ , \qquad \varepsilon_4 = 1 \quad \text{at} \quad t = 0 \tag{n}$$

which means that the unit vectors $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ have been chosen such that $\underline{a}_i = \underline{b}_i$ $(i = 1,2,3)$ at $t = 0$.

Since the differential equations governing ε_1 , ... , ε_4 have time-dependent coefficients, they cannot be solved by simple analytical procedures. However, attacking the physical problem at hand by a different method (see Sec.), and defining a quantity p as

$$p \triangleq [\overline{\omega}_1{}^2 + \overline{\omega}_2{}^2 + (\overline{\omega}_3\ J/I)^2]^{\frac{1}{2}}$$

one can show that ε_1 , ε_2 , ε_3 , and ε_4 are given by

$$\varepsilon_1 = \frac{\sin\ (pt/2)}{p}\ [\overline{\omega}_1 \cos\ (st/2) + \overline{\omega}_2 \sin\ (st/2)]$$

$$\varepsilon_2 = \frac{\sin\ (pt/2)}{p}\ [-\overline{\omega}_1 \sin\ (st/2) + \overline{\omega}_2 \cos\ (st/2)]$$

$$\varepsilon_3 = \overline{\omega}_3(J/Ip) \sin\ (pt/2) \cos\ (st/2) + \cos\ (pt/2) \sin\ (st/2)$$

$$\varepsilon_4 = -\overline{\omega}_3(J/Ip) \sin\ (pt/2) \sin\ (st/2) + \cos\ (pt/2) \cos\ (st/2)$$

and it may be verified that these expressions do, indeed, satisfy the differential equations governing ε_1 , ... , ε_4 as well as the initial conditions stated in Eqs. (n).

1.14 Angular velocity and Rodrigues parameters

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are two dextral sets of orthogonal unit vectors fixed respectively in reference frames or rigid bodies A and B which are moving relative to each other, one can use Eqs. (1.4.9) to associate with each instant of time Rodgrigues parameters ρ_1, ρ_2, and ρ_3; and a Rodrigues vector $\underline{\rho}$ can then be formed by reference to Eq. (1.4.2). The angular velocity of B in A (see Sec. 1.11), expressed in terms of $\underline{\rho}$, is given by

$$\underline{\omega} = \frac{2}{1 + \underline{\rho}^2} \left(\frac{{}^B d\underline{\rho}}{dt} - \underline{\rho} \times \frac{{}^B d\underline{\rho}}{dt} \right) \tag{1}$$

Conversely, if $\underline{\omega}$ is known as a function of time, the Rodrigues parameters can be found by solving the differential equation

$$\frac{{}^B d\underline{\rho}}{dt} = \frac{1}{2} \left(\underline{\omega} + \underline{\rho} \times \underline{\omega} + \underline{\rho}\, \underline{\rho} \cdot \underline{\omega} \right) \tag{2}$$

Equations equivalent to Eqs. (1) and (2) can be formulated in terms of matrices ω, ρ, and $\tilde{\rho}$ defined as

$$\omega \triangleq [\omega_1 \quad \omega_2 \quad \omega_3] \tag{3}$$

$$\rho \triangleq [\rho_1 \quad \rho_2 \quad \rho_3] \tag{4}$$

and

$$\tilde{\rho} \triangleq \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \tag{5}$$

These equations are

$$\omega = \frac{2\dot{\rho}(U + \rho)}{1 + \rho\rho^T} \tag{6}$$

and

$$\dot{\rho} = \frac{1}{2}\,\omega\,(U - \tilde{\rho} + \rho^T\rho) \tag{7}$$

Like its counterparts for the direction cosine matrix and for Euler parameters (see Eqs. (1.10.3) and (1.13.8), Eq. (7) is in general an equation with variable coefficients. Since it is, moreover, nonlinear, one must usually resort to numerical methods to obtain solutions.

<u>Derivations</u>: Using Eqs. (1.3.1), (1.3.3), and (1.4.1), one can express $\underline{\epsilon}$ and ϵ_4 as

$$\underline{\epsilon} = \underline{\rho}(1 + \underline{\rho}^2)^{-1/2}$$

and

$$\epsilon_4 = (1 + \underline{\rho}^2)^{-1/2}$$

respectively. Consequently,

$$\frac{{}^B d\underline{\epsilon}}{dt} = \frac{{}^B d\underline{\rho}}{dt}(1 + \underline{\rho}^2)^{-1/2} - \underline{\rho}(1 + \underline{\rho}^2)^{-3/2}\,\underline{\rho} \cdot \frac{{}^B d\underline{\rho}}{dt}$$

1.14

$$\dot{\epsilon}_4 = -(1 + \underline{\rho}^2)^{-3/2}\, \underline{\rho} \cdot \frac{{}^{B}d\underline{\rho}}{dt}$$

and

$$\underline{\omega} \underset{(1.13.1)}{=} 2\left[\frac{{}^{B}d\underline{\rho}}{dt}(1 + \underline{\rho}^2)^{-1} - \underline{\rho}(1 + \underline{\rho}^2)^{-2}\, \underline{\rho} \cdot \frac{{}^{B}d\underline{\rho}}{dt} + \underline{\rho}(1 + \underline{\rho}^2)^{-2}\, \underline{\rho} \cdot \frac{{}^{B}d\underline{\rho}}{dt} - \underline{\rho} \times \frac{{}^{B}d\underline{\rho}}{dt}(1 + \underline{\rho}^2)^{-1}\right]$$

which is equivalent to Eq. (1).

Cross-multiplication of Eq. (1) with $\underline{\rho}$ yields

$$\underline{\omega} \times \underline{\rho} = \frac{2}{1 + \underline{\rho}^2}\left(\frac{{}^{B}d\underline{\rho}}{dt} \times \underline{\rho} - \underline{\rho}^2 \frac{{}^{B}d\underline{\rho}}{dt} + \underline{\rho}\, \frac{{}^{B}d\underline{\rho}}{dt} \cdot \underline{\rho}\right)$$

$$\underset{(1)}{=} \underline{\omega} - 2\frac{{}^{B}d\underline{\rho}}{dt} + \frac{2}{1 + \underline{\rho}^2}\, \frac{{}^{B}d\underline{\rho}}{dt} \cdot \underline{\rho}\, \underline{\rho}$$

while dot-multiplication produces

$$\underline{\omega} \cdot \underline{\rho} = \frac{2}{1 + \underline{\rho}^2}\, \frac{{}^{B}d\underline{\rho}}{dt} \cdot \underline{\rho}$$

Consequently

$$\underline{\omega} \times \underline{\rho} = \underline{\omega} - 2\frac{{}^{B}d\underline{\rho}}{dt} + \underline{\omega} \cdot \underline{\rho}\, \underline{\rho}$$

in agreement with Eq. (2).

The validity of Eqs. (6) and (7) may be verified by carrying out the indicated matrix multiplications and then comparing the scalar equations corresponding to the matrix equations with the scalar equations corresponding to Eqs. (1) and (2).

Example: The "spin-up" problem for an axially symmetric satellite B can be formulated most simply as follows: Taking the axis of revolution of the inertia ellipsoid of B for the mass center B* of B parallel to $\underline{b}_3$, assuming that B is subjected to the action of a system of forces whose resultant moment about B* is equal to $M\underline{b}_3$, where M is a constant, and letting ω_1, ω_2, and ω_3 have the values

$$\omega_1 = \bar{\omega}_1, \ \omega_2 = \omega_3 = 0 \qquad \text{(a)}$$

at time $t = 0$, determine the orientation of B in an inertial reference frame A for $t > 0$. (The reason for taking ω_2 equal to zero at $t = 0$ is that the unit vectors $\underline{b}_1$ and $\underline{b}_2$ can always be chosen such that $\underline{b}_2$ is perpendicular to $\underline{\omega}$ at $t = 0$, in which case $\omega_2 = \underline{\omega} \cdot b_2 = 0$. As for ω_3, this is taken equal to zero at $t = 0$ because the satellite is presumed to have either no rotational motion or to be tumbling initially, tumbling here referring to a motion such that the angular velocity is perpendicular to the symmetry axis.)

Letting $\underline{I}$ denote the inertia dyadic of B for B*, and defining I and J as

$$I \triangleq \underline{b}_1 \cdot \underline{I} \cdot \underline{b}_1 = \underline{b}_2 \cdot \underline{I} \cdot \underline{b}_2 \qquad \text{(b)}$$

and

$$J \triangleq \underline{b}_3 \cdot \underline{I} \cdot \underline{b}_3 \tag{c}$$

one can use the angular momentum principle to obtain the following differential equations governing ω_1, ω_2, and ω_3:

$$\dot{\omega}_1 = \frac{I - J}{I} \omega_2\omega_3 \tag{d}$$

$$\dot{\omega}_2 = -\frac{I - J}{I} \omega_3\omega_1 \tag{e}$$

$$\dot{\omega}_3 = \frac{M}{J} \tag{f}$$

Since M and J are constants,

$$\omega_3 \underset{(f,a)}{=} \frac{M}{J} t \tag{g}$$

and

$$\dot{\omega}_1 \underset{(d,g)}{=} \frac{I - J}{I} \frac{M}{J} t\, \omega_2 \tag{h}$$

$$\dot{\omega}_2 \underset{(e,g)}{=} -\frac{I - J}{I} \frac{M}{J} t\, \omega_1 \tag{i}$$

The solution of these equations is facilitated by introducing a function ϕ as

$$\phi \triangleq \frac{I - J}{I} \frac{M}{J} \frac{t^2}{2}$$

Then

$$\dot{\omega}_1 \underset{(h,j)}{=} \dot{\phi}\omega_2 \qquad (k)$$

$$\dot{\omega}_2 \underset{(i,j)}{=} -\dot{\phi}\omega_1 \qquad (l)$$

or

$$\frac{d\omega_1}{d\phi} \underset{(k)}{=} \omega_2 \qquad (m)$$

$$\frac{d\omega_2}{d\phi} \underset{(l)}{=} -\omega_1 \qquad (n)$$

so that

$$\frac{d^2\omega_1}{d\phi^2} + \omega_1 \underset{(m,n)}{=} 0 \qquad (o)$$

$$\omega_1 = C_1 \sin\phi + C_2 \cos\phi \qquad (p)$$

and

$$\omega_2 \underset{(m,p)}{=} C_1 \cos\phi - C_2 \sin\phi \qquad (q)$$

where C_1 and C_2 are constants which can be evaluated by noting that ϕ (see Eq. (j)) vanishes at $t = 0$. That is,

$$\bar{\omega}_1 \underset{(a,p)}{=} C_2 \qquad (r)$$

and

$$0 \underset{(a,q)}{=} C_1 \tag{s}$$

Consequently

$$\omega_1 \underset{(p)}{=} \bar{\omega}_1 \cos \phi \tag{t}$$

and

$$\omega_2 \underset{(q)}{=} -\bar{\omega}_1 \sin \phi \tag{u}$$

Equations governing the Rodrigues parameters ρ_1, ρ_2, and ρ_3 can now be formulated by referring to Eqs. (3), (4), (5), and (7) to obtain

$$2\dot{\rho}_1 = \bar{\omega}_1 \cos \phi \left(1 + \rho_1^2\right) - \bar{\omega}_1 \sin \phi \, (\rho_1\rho_2 - \rho_3) + \frac{M}{J} t \, (\rho_3\rho_1 + \rho_2) \tag{v}$$

$$2\dot{\rho}_2 = \bar{\omega}_1 \cos \phi \, (\rho_1\rho_2 + \rho_3) - \bar{\omega}_1 \sin \phi \left(1 + \rho_2^2\right) + \frac{M}{J} t \, (\rho_2\rho_3 - \rho_1) \tag{w}$$

$$2\dot{\rho}_3 = \bar{\omega}_1 \cos \phi \, (\rho_3\rho_1 - \rho_2) - \bar{\omega}_1 \sin \phi \, (\rho_2\rho_3 + \rho_1) + \frac{M}{J} t \left(1 + \rho_3^2\right) \tag{x}$$

and, if $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are chosen such that $\underline{a}_i = \underline{b}_i$ $(i = 1,2,3)$ at $t = 0$, then ρ_1, ρ_2, and ρ_3 must satisfy the initial conditions

$$\rho_i(0) = 0 \qquad (i = 1,2,3) \tag{y}$$

Suppose now that one wishes to study the behavior of the symmetry axis of B, say for $0 \leq \bar{\omega}_1 t \leq 10.0$, by plotting the angle θ between this axis and the line fixed in A with which the symmetry axis coincides initially. Once the dimensionless parameters J/I and $M/J\bar{\omega}_1^{\,2}$ have been specified, ρ_1, ρ_2, and ρ_3 can be evaluated by integrating Eqs. (v) - (x) numerically, and θ is then given by

$$\theta = \text{arc cos } (\underline{a}_3 \cdot \underline{b}_3) \underset{(1.2.2,\ 1.2.3)}{=} C_{33} \underset{(1.4.4)}{=} \frac{1 - \rho_1^{\,2} - \rho_2^{\,2} + \rho_3^{\,2}}{1 + \rho_1^{\,2} + \rho_2^{\,2} + \rho_3^{\,2}} \tag{z}$$

Table 1 shows values of ρ_1, ρ_2, ρ_3 and θ obtained in this way for $J/I = 0.5$ and $M/J\bar{\omega}_1^{\,2} = 0.1$. The largest value of $\bar{\omega}_1 t$ appearing in the table is 3.0, rather than 10.0, because during integration from 3.0 to 3.5 the values of ρ_1, ρ_2, and ρ_3 became so large that the integration could not be continued. To overcome this obstacle, Eqs. (v) - (z) were replaced with (see Eqs. (1.13.8))

$$2\dot{\epsilon}_1 = \bar{\omega}_1 \cos \phi \ \epsilon_4 + \bar{\omega}_1 \sin \phi \ \epsilon_3 + \frac{M}{J} t \ \epsilon_2$$

$$2\dot{\epsilon}_2 = \bar{\omega}_1 \cos \phi \ \epsilon_3 - \bar{\omega}_1 \sin \phi \ \epsilon_4 - \frac{M}{J} t \ \epsilon_1$$

$$2\dot{\epsilon}_3 = -\bar{\omega}_1 \cos \phi - \epsilon_2 - \bar{\omega}_1 \sin \phi \ \epsilon_1 + \frac{M}{J} t \ \epsilon_4$$

$$\epsilon_1(0) = \epsilon_2(0) = \epsilon_3(0) = 0, \ \epsilon_4(0) = 1$$

and

$$\theta \underset{(1.3.14)}{=} \text{arc cos} \left(1 - 2\epsilon_1^{\,2} - 2\epsilon_2^{\,2}\right)$$

Table 1

$\bar{\omega}_1 t$	ρ_1	ρ_2	ρ_3	θ(deg)
0.0	0.00	0.00	0.00	0
0.5	0.26	-0.00	0.00	29
1.0	0.55	-0.01	0.03	57
1.5	0.93	-0.04	0.06	86
2.0	1.56	-0.11	0.13	115
2.5	3.06	-0.33	0.27	143
3.0	16.94	-2.69	1.41	169

respectively, and a numerical integration of these equations, performed without difficulty because $-1 \leq \epsilon_i \leq 1$ $(i = 1, \ldots, 4)$, produced the values listed in Table 2. These results not only permit one to plot θ versus $\bar{\omega}_1 t$, as has been done in Fig. 1.14.1, but they indicate quite clearly why the numerical solution of Eqs. (v) - (x) could not proceed smoothly: ϵ_4 changes sign between $\bar{\omega}_1 t = 3.0$ and $\bar{\omega}_1 t = 3.5$, and again between $\bar{\omega}_1 t = 8.5$ and $\bar{\omega}_1 t = 9.0$, whereas $\epsilon_i (i = 1,2,3)$ do not change sign in these intervals. Hence ϵ_4 vanishes at two points at which $\epsilon_i (i = 1,2,3)$ do not vanish, and since

$$\rho_i \underset{(1.4.3)}{=} \frac{\epsilon_i}{\epsilon_4} \quad (i = 1,2,3)$$

the Rodrigues parameters become infinite at these two points.

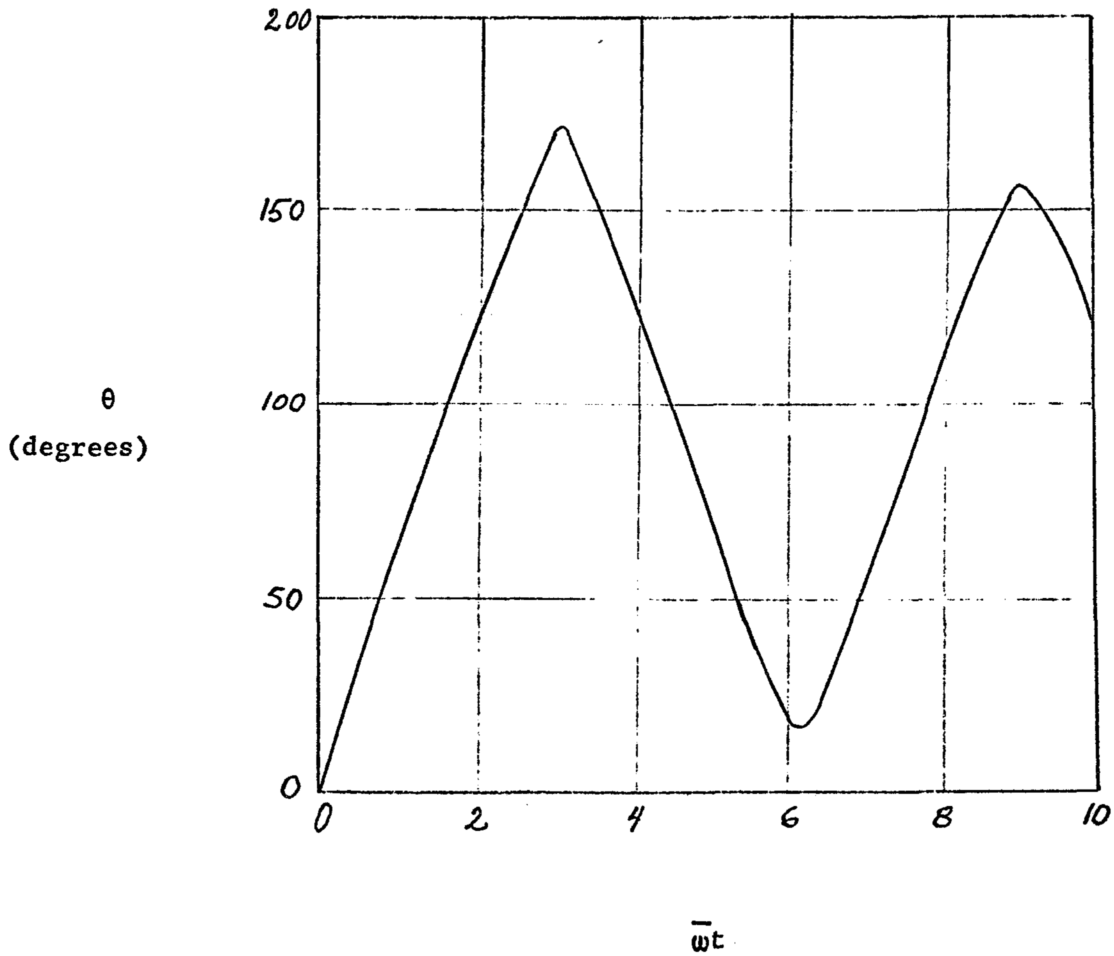


Figure 1.14.1

Table 2

$\bar{\omega}_1 t$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	θ(deg)
0.0	0.00	0.00	0.00	0.00	0
0.5	0.25	0.00	0.01	0.97	29
1.0	0.48	-0.01	0.02	0.88	57
1.5	0.68	-0.03	0.05	0.73	86
2.0	0.84	-0.06	0.07	0.54	115
2.5	0.94	-0.10	0.08	0.31	143
3.0	0.98	-0.16	0.08	0.06	169
3.5	0.96	-0.21	0.06	-0.20	157
4.0	0.86	-0.26	0.00	-0.43	129
4.5	0.71	-0.30	-0.01	-0.64	100
5.0	0.50	-0.30	-0.18	-0.79	72
5.5	0.26	-0.26	-0.30	-0.88	44
6.0	0.02	-0.17	-0.41	-0.89	20
6.5	-0.22	-0.04	-0.51	-0.83	26
7.0	-0.42	0.13	-0.57	-0.70	52
7.5	-0.55	0.33	-0.58	-0.50	80
8.0	-0.61	0.53	-0.51	-0.28	108
8.5	-0.59	0.71	-0.37	-0.06	136
9.0	-0.49	0.84	-0.17	0.14	155
9.5	-0.33	0.90	0.09	0.28	146
10.0	-0.13	0.86	0.36	0.34	120

1.15 Indirect determination of angular velocity

When a rigid body B can be observed from a vantage point fixed in a reference frame A, the angular velocity $\underline{\omega}$ of B in A can be determined by using Eq. (1.11.4). If observations permitting such a direct evaluation of $\underline{\omega}$ cannot be made, it may, nevertheless, be possible to find $\underline{\omega}$. This is the case, for example, when two vectors, say $\underline{p}$ and $\underline{q}$, can each be observed from a vantage point fixed in A as well as from one fixed in B, for $\underline{\omega}$ can then be found by using the relationship

$$\underline{\omega} = \frac{\left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \times \left(\frac{{}^{A}d\underline{q}}{dt} - \frac{{}^{B}d\underline{q}}{dt}\right)}{\left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \cdot \underline{q}} \tag{1}$$

To carry out the algebraic operations indicated in this equation, one must be able to express all vectors in a common basis. This can be accomplished by using Eq. (1.2.9) after forming a direction cosine matrix by reference to Eq. (1.5.3).

<u>Derivation</u>: From Eq. (1.11.8),

$$\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt} = \underline{\omega} \times \underline{p} \tag{a}$$

and

$$\frac{{}^{A}d\underline{q}}{dt} - \frac{{}^{B}d\underline{q}}{dt} = \underline{\omega} \times \underline{q} \tag{b}$$

Hence

$$\left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \times \left(\frac{{}^{A}d\underline{q}}{dt} - \frac{{}^{B}d\underline{q}}{dt}\right) \underset{(b)}{=} \left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \times (\underline{\omega} \times \underline{q})$$

$$= \left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \cdot \underline{q}\,\underline{\omega} - \left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \cdot \underline{\omega}\,\underline{q}$$

$$\underset{(a)}{=} \left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \cdot \underline{q}\,\underline{\omega} - \underline{\omega} \times \underline{p} \cdot \underline{\omega}\,\underline{q}$$

$$= \left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right) \cdot \underline{q}\,\underline{\omega} + 0$$

and, solving for $\underline{\omega}$, one arrives at Eq. (1).

Example: Observations of two stars, P and Q, are made simultaneously from two space vehicles, A and B, these observations consisting of measuring the angles ϕ and ψ shown in Fig. 1.15.1, where O represents either a point fixed in A or a point fixed in B, R is either P or Q, and $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ are orthogonal unit vectors forming a dextral set fixed either in A or in B.

For a certain instant, the angles and their first time-derivatives are found to have the values shown in Table 1. The angular velocity $\underline{\omega}$ of B in A at that instant is to be determined.

The situation under consideration is the same as that discussed in the Example in Sec. 1.5. Hence, if $\underline{v}$ is any vector and ${}^{A}\underline{v}$ and ${}^{B}\underline{v}$

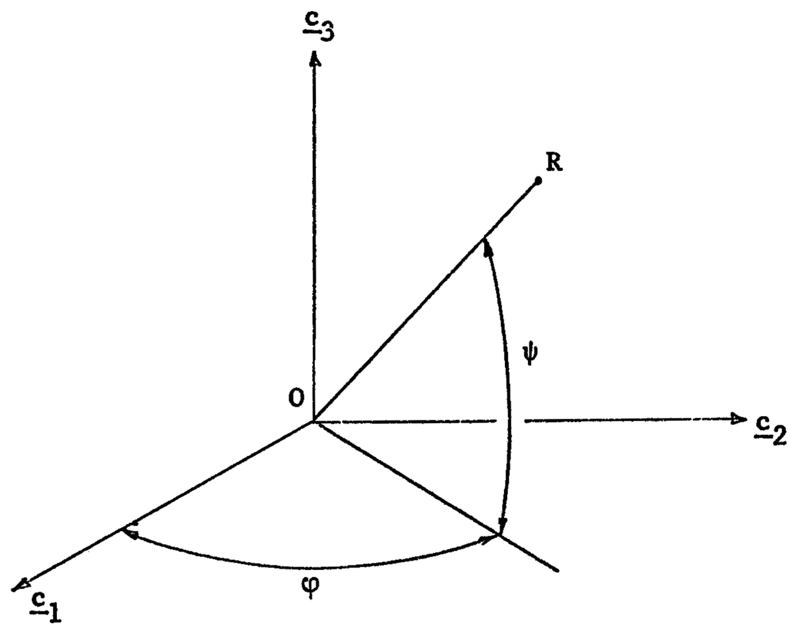


Figure 1.15.1

Table 1

ϕ and ψ in degrees, $\dot{\phi}$ and $\dot{\psi}$ in radians per second

	P				Q			
	ϕ	ψ	$\dot{\phi}$	$\dot{\psi}$	ϕ	$\dot{\psi}$	ϕ	$\dot{\psi}$
$\underline{c}_i = \underline{a}_i$	90	45	-2	1	30	0	0	$\frac{1}{2} - \sqrt{3}$
$\underline{c}_i = \underline{b}_i$	135	0	0	$-3\sqrt{3}/2$	90	60	$3\sqrt{3}$	0

are row matrices having $\underline{v} \cdot \underline{a}_i$ and $\underline{v} \cdot \underline{b}_i$ $(i = 1,2,3)$ as elements, then

$$B_v \underset{(1.2.9)}{=} A_v \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{c}$$

Furthermore, if $\underline{R}$ is again defined as a unit vector directed from O toward R (see Fig. 1.15.1), then

$$\underline{R} = \cos\psi \cos\phi\, \underline{c}_1 + \cos\psi \sin\phi\, \underline{c}_2 + \sin\psi\, \underline{c}_3 \tag{d}$$

and the first time-derivative of $\underline{R}$ is a reference frame C in which $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ are fixed is given by

$$\begin{aligned} \frac{{}^C d\underline{R}}{dt} \underset{(d)}{=} & - (\sin\psi \cos\phi\, \dot{\psi} + \cos\psi \sin\phi\, \dot{\phi})\underline{c}_1 \\ & - (\sin\psi \sin\phi\, \dot{\psi} - \cos\psi \cos\phi\, \dot{\phi})\underline{c}_2 \\ & + \cos\psi\, \dot{\psi}\, \underline{c}_3 \end{aligned} \tag{e}$$

Consequently, letting $\underline{p}$ and $\underline{q}$ be unit vectors directed from O toward P and Q respectively, and referring to Table 1, one can express the time-derivatives of $\underline{p}$ and $\underline{q}$ in A as

$$\frac{{}^{A}d\underline{p}}{dt} \underset{(e)}{=} \sqrt{2}\,\underline{a}_1 - \frac{\sqrt{2}}{2}\,\underline{a}_2 + \frac{\sqrt{2}}{2}\,\underline{a}_3 \tag{f}$$

and

$$\frac{{}^{A}d\underline{q}}{dt} \underset{(e)}{=} \left(\frac{1}{2} - \sqrt{3}\right)\underline{a}_3 \tag{g}$$

Next, use of Eq. (c) permits one to express these derivatives in terms of $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$, as indicated in lines 1 and 2 of Table 2; and lines 3 and 4 are formed similarly. Lines 5, 6, and 7 can then be formed by purely algebraic operations, and the scalar product appearing in the denominator of the right-hand member of Eq. (1) is given by (see line 2 of Table 2 in the Example in Sec. 1.5)

$$\left(\frac{{}^{A}d\underline{p}}{dt} - \frac{{}^{B}d\underline{p}}{dt}\right)\cdot\underline{q} = -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) + \frac{5\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{5\sqrt{3}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}$$

Consequently

$$\underline{\omega} \underset{(1)}{=} \frac{\frac{5\sqrt{2}}{2}\left(\frac{5\sqrt{3}}{2} - \frac{1}{2}\right)\underline{b}_2 + \frac{\sqrt{2}}{2}\left(\frac{5\sqrt{3}}{2} - \frac{1}{2}\right)\underline{b}_3}{\left(\frac{5\sqrt{3}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}}$$

$$= 5\underline{b}_2 + \underline{b}_3 \text{ rad/sec.}$$

Table 2

Vectors appearing in Eq. (1)

Line	Vector	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}_3$
1	$\frac{{}^{A}d\underline{p}}{dt}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\sqrt{2}$
2	$\frac{{}^{A}d\underline{q}}{dt}$	$\sqrt{3}-\frac{1}{2}$	0	0
3	$\frac{{}^{B}d\underline{p}}{dt}$	0	0	$-\frac{3\sqrt{2}}{2}$
4	$\frac{{}^{B}d\underline{q}}{dt}$	$-\frac{3\sqrt{3}}{2}$	0	0
5	$\frac{{}^{A}d\underline{p}}{dt}-\frac{{}^{B}d\underline{p}}{dt}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{5\sqrt{2}}{2}$
6	$\frac{{}^{A}d\underline{q}}{dt}-\frac{{}^{B}d\underline{q}}{dt}$	$\frac{5\sqrt{3}}{2}-\frac{1}{2}$	0	0
7	$\left(\frac{{}^{A}d\underline{p}}{dt}-\frac{{}^{B}d\underline{p}}{dt}\right)\times\left(\frac{{}^{A}d\underline{q}}{dt}-\frac{{}^{B}d\underline{q}}{dt}\right)$	0	$\frac{5\sqrt{2}}{2}\left(\frac{5\sqrt{3}}{3}-\frac{1}{2}\right)$	$\frac{\sqrt{2}}{2}\left(\frac{5\sqrt{3}}{2}-\frac{1}{2}\right)$

1.16 Auxiliary reference frames

The angular velocity of a rigid body B in a reference frame A (see Sec. 1.11) can be expressed in the following form involving n auxiliary reference frames $A_1, \ldots, A_n$:

$$^{A}\underline{\omega}^{B} = {}^{A}\underline{\omega}^{A_1} + {}^{A_1}\underline{\omega}^{A_2} + \ldots + {}^{A_{n-1}}\underline{\omega}^{A_n} + {}^{A_n}\underline{\omega}^{B} \tag{1}$$

This relationship is particularly useful when each term in the right-hand member represents the angular velocity of a body performing a motion of simple rotation (see Sec. 1.1) and can, therefore, be expressed as in Eq. (1.11.5).

Derivation: For any vector $\underline{c}$ fixed in B,

$$\frac{^{A}d\underline{c}}{dt} \underset{(1.11.9)}{=} {}^{A}\underline{\omega}^{B} \times \underline{c} \tag{a}$$

$$\frac{^{A_1}d\underline{c}}{dt} \underset{(1.11.9)}{=} {}^{A_1}\underline{\omega}^{B} \times \underline{c} \tag{b}$$

and

$$\frac{^{A}d\underline{c}}{dt} \underset{(1.11.8)}{=} \frac{^{A_1}d\underline{c}}{dt} + {}^{A}\underline{\omega}^{A_1} \times \underline{c} \tag{c}$$

so that

$$^{A}\underline{\omega}^{B} \times \underline{c} \underset{(a,b,c)}{=} {}^{A}\underline{\omega}^{A_1} \times \underline{c} + {}^{A_1}\underline{\omega}^{B} \times \underline{c} \tag{d}$$

or, since this equation is satisfied for <u>every</u> $\underline{c}$ fixed in B,

$$^{A}\underline{\omega}^{B} = {}^{A}\underline{\omega}^{A_1} + {}^{A_1}\underline{\omega}^{B} \tag{e}$$

which shows that Eq. (1) is valid for $n = 1$. Proceeding similarly, one can verify that

$$^{A_1}\underline{\omega}^{B} = {}^{A_1}\underline{\omega}^{A_2} + {}^{A_2}\underline{\omega}^{B} \tag{f}$$

and substitution into Eq. (e) then yields

$$^{A}\underline{\omega}^{B} = {}^{A}\underline{\omega}^{A_1} + {}^{A_1}\underline{\omega}^{A_2} + {}^{A_2}\underline{\omega}^{B}$$

which is Eq. (1) for $n = 2$. The validity of Eq. (1) for any value of n can thus be established by applying this procedure a sufficient number of times.

<u>Example</u>: In Fig. 1.16.1, θ, ϕ, and ψ designate angles used to describe the orientation of a rigid cone B in a reference frame A. These angles are formed by lines described as follows: L_1 and L_2 are perpendicular to each other and fixed in A; L_3 is the axis of symmetry of B; L_4 is perpendicular to L_2 and intersects L_2 and L_3; L_5 is perpendicular to L_3 and fixed in B; and L_7 is perpendicular to L_2 and L_4. To find the angular velocity of B in A, one can designate as A_1 a reference frame in which L_2, L_4 and L_7 are fixed, and as A_2 a reference frame in which L_3, L_5, and L_7 are fixed, observing

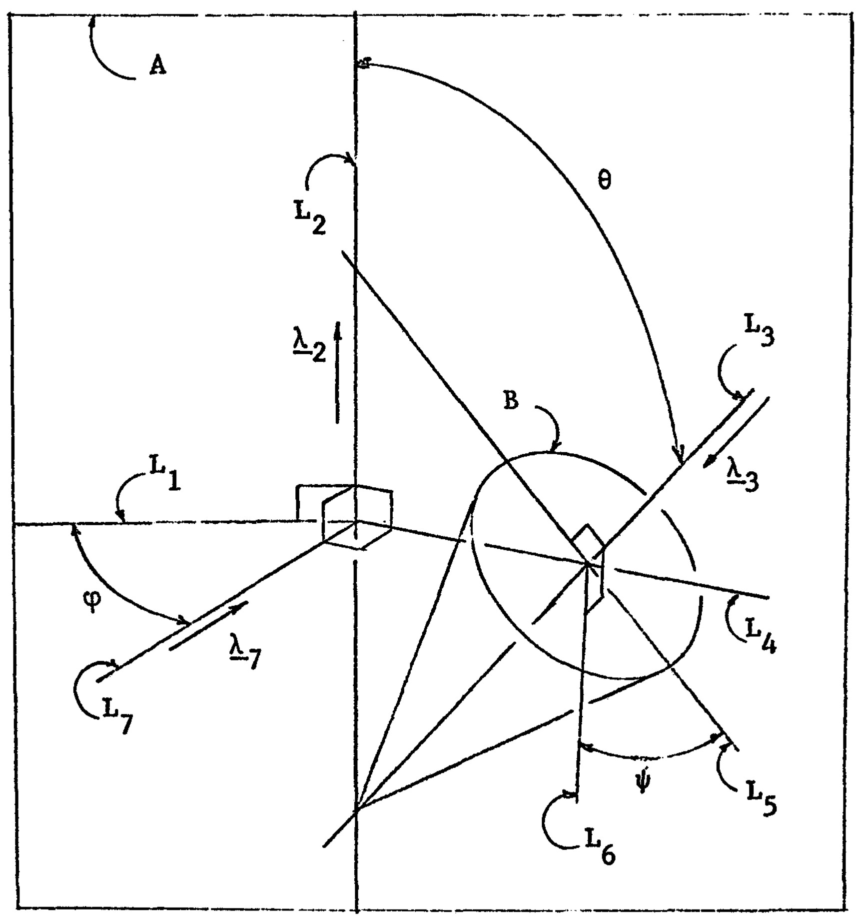

A
θ
L_2
$\underline{\lambda}_2$
L_3
B
$\underline{\lambda}_3$
L_1
L_4
φ
$\underline{\lambda}_7$
L_7
ψ
L_5
L_6

Figure 1.16.1

that L_2 is then fixed both in A and A_1, L_7 is fixed both in A_1 and A_2, and L_3 is fixed both in A_2 and B, so that, in accordance with Eq. (1.11.5),

$$^{A}\underline{\omega}^{A_1} = \dot{\phi}\underline{\lambda}_2, \quad ^{A_1}\underline{\omega}^{A_2} = \dot{\theta}\underline{\lambda}_7, \quad ^{A_2}\underline{\omega}^{B} = \dot{\psi}\underline{\lambda}_3$$

where $\underline{\lambda}_2$, $\underline{\lambda}_3$, and $\underline{\lambda}_7$ are unit vectors directed as shown in Fig. 1.16.1. It then follows immediately that

$$^{A}\underline{\omega}^{B} \underset{(1)}{=} \dot{\phi}\underline{\lambda}_2 + \dot{\theta}\underline{\lambda}_7 + \dot{\psi}\underline{\lambda}_3$$

1.17 Angular velocity and orientation angles

When the orientation of a rigid body B in a reference frame A is described by specifying the time dependence of orientation angles θ_1, θ_2, and θ_3 (see Sec. 1.7), the angular velocity of B in A (see Sec. 1.11) can be found by using the relationship

$$[\omega_1\ \omega_2\ \omega_3] = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3]\ M \tag{1}$$

where M is a 3 × 3 matrix whose elements are functions of θ_1, θ_2, and θ_3. Conversely, if ω_1, ω_2, and ω_3 are known as functions of time, then θ_1, θ_2, and θ_3 can be evaluated by solving the differential equations

$$[\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3] = [\omega_1\ \omega_2\ \omega_3]\ M^{-1} \tag{2}$$

For space-three-vector angles, the matrices M and M^{-1} are

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix} \tag{3}$$

and

$$M^{-1} = \frac{1}{c_2}\begin{bmatrix} c_2 & 0 & 0 \\ s_1s_2 & c_1c_2 & s_1 \\ c_1s_2 & -s_1c_2 & c_1 \end{bmatrix} \tag{4}$$

For body-three-vector angles,

$$M = \begin{bmatrix} c_2c_3 & -c_2s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

and

$$M^{-1} = \frac{1}{c_2}\begin{bmatrix} c_3 & c_2s_3 & -s_2c_3 \\ -s_3 & c_2c_3 & s_2s_3 \\ 0 & 0 & c_2 \end{bmatrix} \tag{6}$$

For space-two-vector angles,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ c_2 & s_1s_2 & c_1s_2 \end{bmatrix} \tag{7}$$

and

$$M^{-1} = \frac{1}{s_2}\begin{bmatrix} s_2 & 0 & 0 \\ -s_1c_2 & c_1s_2 & s_1 \\ -c_1c_2 & -s_1s_2 & c_1 \end{bmatrix} \tag{8}$$

Finally, for body-two-vector angles,

$$M = \begin{bmatrix} c_2 & s_2s_3 & s_2c_3 \\ 0 & c_3 & -s_3 \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

and

$$M^{-1} = \frac{1}{s_2}\begin{bmatrix} 0 & 0 & s_2 \\ s_3 & s_2c_3 & -c_2s_3 \\ c_3 & -s_2s_3 & -c_2c_3 \end{bmatrix} \tag{10}$$

When c_2 vanishes, M as given by Eq. (3) or by Eq. (5) is a singular matrix, and M^{-1} is thus undefined. Hence, given ω_1, ω_2, and ω_3 one cannot use Eq. (2) to determine $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ if θ_1, θ_2, and θ_3 are three-vector angles and $c_2 = 0$. Similarly, if θ_1, θ_2, and θ_3 are two-vector angles, Eq. (2) involves an undefined matrix when s_2 is equal to zero.

When the angles and unit vectors employed in an analysis are denoted by symbols other than those used in connection with Eqs. (1) - (10), appropriate replacements for these equations can be obtained directly from Eqs. (1) - (10) whenever the angles have been identified as regards type, that is, as being space-three-vector angles, body-three-vector angles, etc. Suppose, for example, that in the course of an analysis involving the cone shown in Fig. 1.16.1, and previously considered in the Example in Sec. 1.16, unit vectors $\underline{b}_x$, $\underline{b}_y$ and $\underline{b}_z$, fixed in B as shown in Fig. 1.17.1, have been introduced, and it is now desired to find ω_x, ω_y, and $\underline{\omega}_z$, defined as

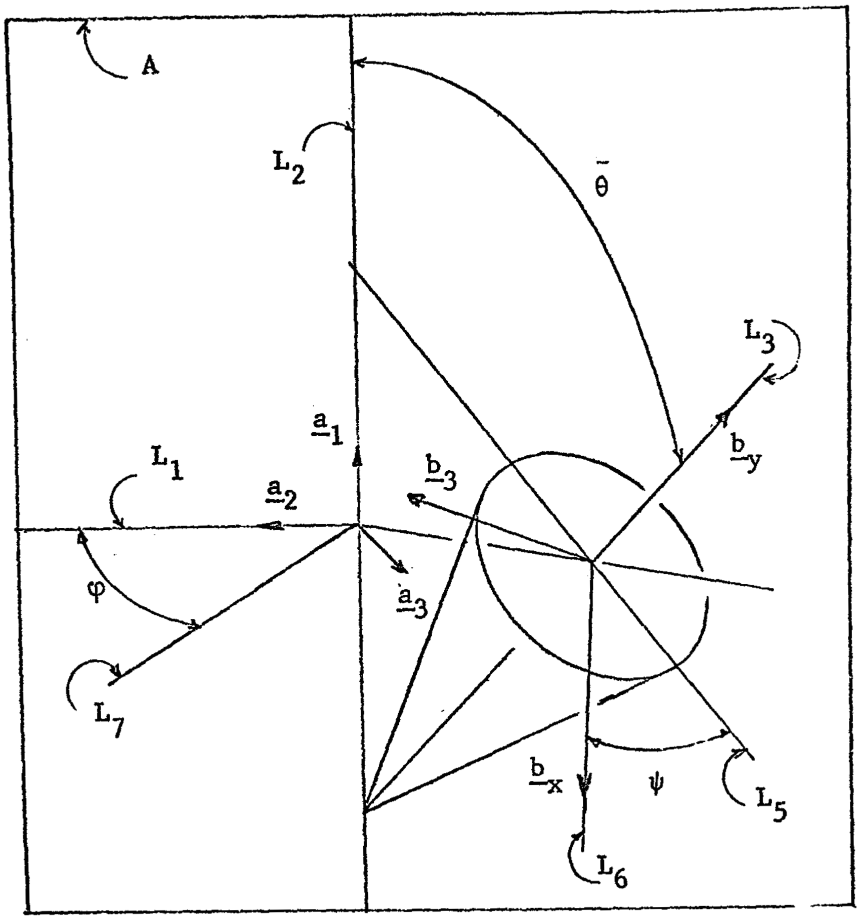

A
L_2
$\bar{\theta}$
L_3
$\underline{a}_1$
$\underline{b}_y$
L_1
$\underline{b}_3$
$\underline{a}_2$
φ
$\underline{a}_3$
L_7
$\underline{b}_x$
ψ
L_5
L_6

Figure 1.17.1

$$\omega_x \triangleq \underline{\omega} \cdot \underline{b}_x, \quad \omega_y \triangleq \underline{\omega} \cdot \underline{b}_y, \quad \omega_z \triangleq \underline{\omega} \cdot \underline{b}_z$$

where $\underline{\omega}$ denotes the angular velocity of B in A. This can be done easily by regarding ϕ, θ, and ψ as body-two-vector angles, that is, by introducing unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ as shown in Fig. 1.17.1, defining $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ as

$$\underline{b}_1 \triangleq \underline{b}_y, \quad \underline{b}_2 \triangleq \underline{b}_z, \quad \underline{b}_3 \triangleq \underline{b}_x$$

and taking

$$\theta_1 = \phi, \quad \theta_2 = -\theta, \quad \theta_3 = -\psi$$

For it then follows immediately from Eqs. (1) and (9) that

$$[\omega_y \ \omega_z \ \omega_x] = [\dot{\phi} \ \ -\dot{\theta} \ \ -\dot{\psi}] \begin{bmatrix} \cos\theta & \sin\theta\sin\psi & -\sin\theta\cos\psi \\ 0 & \cos\psi & \sin\psi \\ 1 & 0 & 0 \end{bmatrix}$$

so that

$$\omega_x = -\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_y = \dot{\phi}\cos\theta - \dot{\psi}$$

$$\omega_z = \dot{\phi}\sin\theta\sin\psi - \dot{\theta}\cos\psi$$

<u>Derivations</u>: From Eqs. (1.10.5) and (1.7.1)

$$\omega_1 = (c_1 s_2 c_3 + s_3 s_1) \frac{d}{dt} (s_1 s_2 c_3 - s_3 c_1)$$

$$+ (c_1 s_2 s_3 - c_3 s_1) \frac{d}{dt} (s_1 s_2 s_3 + c_3 c_1)$$

$$+ c_1 c_2 \frac{d}{dt} (s_1 c_2)$$

$$= \dot{\theta}_1 - \dot{\theta}_3 s_2$$

Similarly, from Eqs. (1.10.6) and (1.7.1)

$$\omega_2 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 c_2$$

and from Eqs. (1.10.7) and (1.7.1)

$$\omega_3 = - \dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$$

These three equations are the three scalar equations corresponding to Eq. (1) when M is given by Eq. (3).

Eq. (2) follows from Eq. (1) and from the definition of the inverse of a matrix; and the validity of Eq. (4) may be established by noting that the product of the right-hand members of Eqs. (3) and (4) is equal to U, the unit matrix. Proceeding similarly, but using Eq. (1.7.11), (1.7.21), or (1.7.31) in place of Eq. (1.7.1), and Eqs. (5) and (6), Eqs. (7) and (8), or Eqs. (9) and (10) in place of Eqs. (3) and (4), one can demonstrate the validity of Eqs. (5) - (10).

<u>Example</u>: Fig. 1.17.2 shows the gyroscopic system previously discussed in Sec. 1.7, where it was mentioned that one may wish to employ

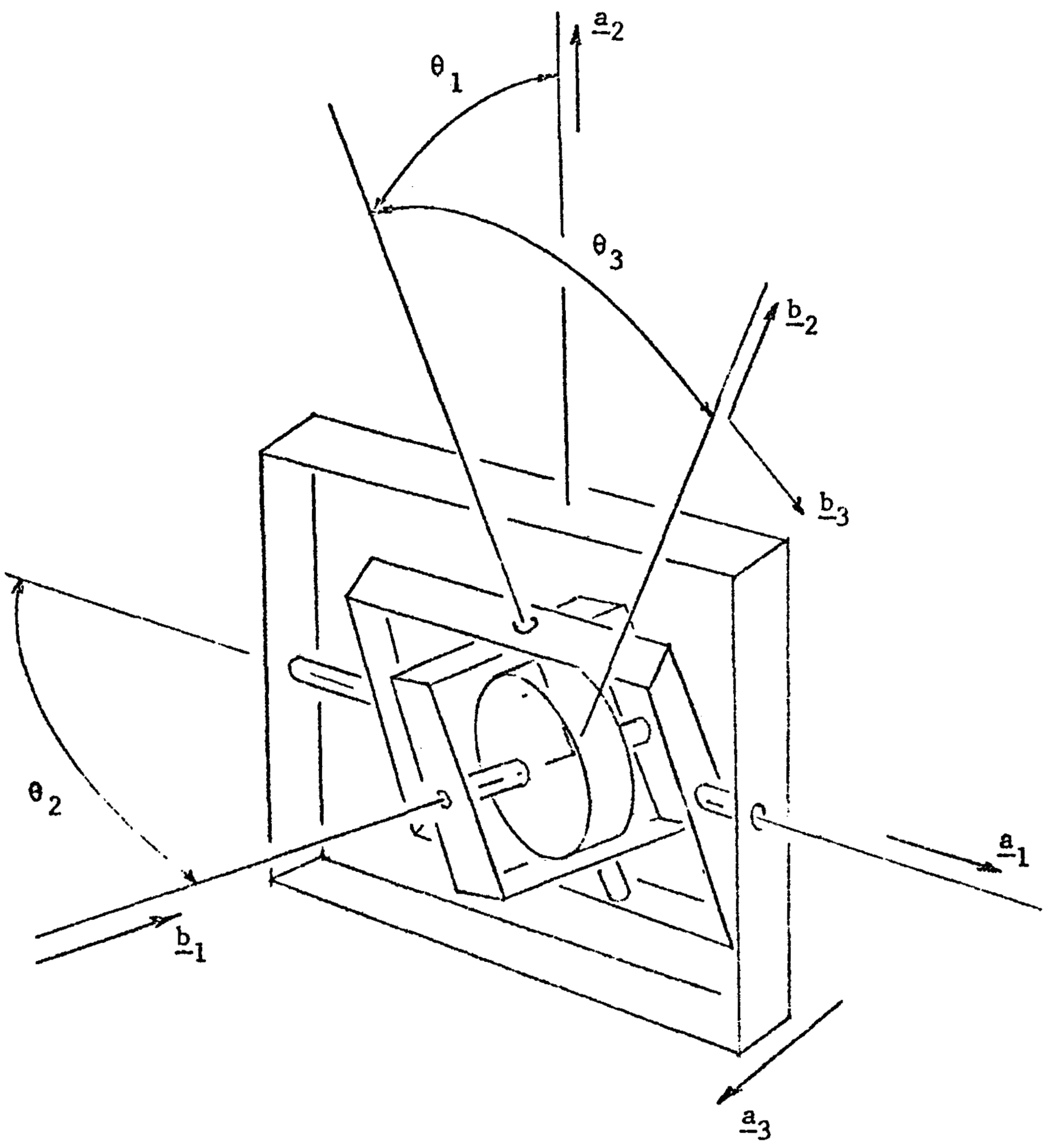

$\underline{a}_2$
θ_1
θ_3
$\underline{b}_2$
$\underline{b}_3$
θ_2
$\underline{a}_1$
$\underline{b}_1$
$\underline{a}_3$

Figure 1.17.2

space-three-vector angles ϕ_1, ϕ_2, and ϕ_3, as well as the body-two-vector angles θ_1, θ_2, and θ_3 shown in Fig. 1.17.2, when analyzing motions during which θ_2 becomes small or equal to zero. Given θ_i and $\dot{\theta}_i$ (i = 1,2,3), one must then be able to evaluate ϕ_i and $\dot{\phi}_i$ (i = 1,2,3).

Suppose that, as in the Example in Sec. 1.7, $\theta_1 = 30°$, $\theta_2 = 45°$, and $\theta_3 = 60°$ at a certain instant and that, furthermore $\dot{\theta}_1 = 1.00$, $\dot{\theta}_2 = 2.00$, $\dot{\theta}_3 = 3.00$ rad/sec. What are the value of $\dot{\phi}_1$, $\dot{\phi}_2$, and $\dot{\phi}_3$ at this instant?

From Eqs. (2) and (4),

$$[\dot{\phi}_1 \ \dot{\phi}_2 \ \dot{\phi}_3] = \frac{[\omega_1 \ \omega_2 \ \omega_3]}{\cos \phi_2} \begin{bmatrix} \cos \phi_2 & 0 & 0 \\ \sin \phi_1 \sin \phi_2 & \cos \phi_1 \cos \phi_2 & \sin \phi_1 \\ \cos \phi_1 \sin \phi_2 & -\sin \phi_1 \cos \phi_2 & \cos \phi_1 \end{bmatrix}$$

or, using the values of ϕ_1 and ϕ_2 found previously,

$$[\dot{\phi}_1 \ \dot{\phi}_2 \ \dot{\phi}_3] = \frac{[\omega_1 \ \omega_2 \ \omega_2]}{0.791} \begin{bmatrix} 0.791 & 0 & 0 \\ 0.603 & -0.138 & 0.985 \\ -0.107 & -0.780 & -0.174 \end{bmatrix}$$

Now, from Eqs. (1) and (9),

$$[\omega_1\ \omega_2\ \omega_3] = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3] \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \sin\theta_3 & \sin\theta_2 \cos\theta_3 \\ 0 & \cos\theta_2 & -\sin\theta_3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [1.00 \quad 2.00 \quad 3.00] \begin{bmatrix} 0.707 & 0.612 & 0.354 \\ 0 & 0.500 & -0.866 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [3.707 \quad 1.612 \quad -1.378]$$

Hence

$$[\dot{\phi}_1\ \dot{\phi}_2\ \dot{\phi}_3] = \frac{[3.707 \quad 1.612 \quad -1.378]}{0.791} \begin{bmatrix} 0.791 & 0 & 0 \\ 0.603 & -0.138 & 0.985 \\ -0.107 & -0.780 & -0.174 \end{bmatrix}$$

$$= [5.12 \quad 1.08 \quad 2.41]$$

and

$$\dot{\phi}_1 = 5.12, \quad \dot{\phi}_2 = 1.08, \quad \dot{\phi}_3 = 2.41 \text{ rad/sec.}$$

1.18 Slow, small rotational motions

If $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are two dextral sets of orthogonal unit vectors fixed respectively in reference frames or rigid bodies A and B which are moving relative to each other, one can use Eqs. (1.3.18) and (1.3.20) to associate with each instant of time an angle θ, and the motion is called a slow, small rotational motion when all terms of second or higher degree in θ and $\dot{\theta}$ play a negligible role in an analysis of the motion. Under these circumstances, a number of the relationships discussed previously can be replaced with simpler ones. Specifically, in place of Eqs. (1.13.1) and (1.13.3) one may then use

$$\underline{\omega} = 2\,\frac{{}^{B}d\underline{\epsilon}}{dt} \tag{1}$$

and

$$\epsilon_4 = 1 \tag{2}$$

Eqs. (1.14.1) and (1.14.2) can be replaced with

$$\underline{\omega} = 2\,\frac{{}^{B}d\underline{\rho}}{dt} \tag{3}$$

and, if θ_1, θ_2, and θ_3 are chosen such that terms of second or higher degree in θ_i and/or $\dot{\theta}_j (i,j = 1,2,3)$ are negligible, then Eq. (1.7.1) together with Eq. (1.17.3) or Eq. (1.17.5) leads to

$$[\omega_1\ \omega_2\ \omega_3] = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3] \tag{4}$$

which shows that it does not matter whether one uses space-three-vector angles or body-three-vector angles when dealing with small, slow rotational motions.

Derivations: From Eqs. (1.8.3) and (1.8.4)

$$\underline{\epsilon} = \frac{1}{2}\underline{\lambda}\theta$$

and

$$\epsilon_4 = 1$$

Hence

$$\frac{{}^B d\underline{\epsilon}}{dt} = \frac{1}{2}\left(\frac{{}^B d\underline{\lambda}}{dt}\theta + \underline{\lambda}\dot{\theta}\right)$$

and, substituting into Eq. (1.13.1) and retaining only terms of first degree in θ and $\dot{\theta}$, one obtains

$$\underline{\omega} = 2\left[\frac{1}{2}\left(\frac{{}^B d\underline{\lambda}}{dt}\theta + \underline{\lambda}\dot{\theta}\right)\right] = 2\,\frac{{}^B d\underline{\rho}}{dt}$$

in agreement with Eq. (1). Eq. (2) is the same as Eq. (1.8.4).

Eq. (3) follows immediately from Eq. (1), since $\underline{\rho}$ and $\underline{\epsilon}$ are equal to each other to the order of approximation under consideration, as is apparent from Eqs. (1.8.3) and (1.8.5). Finally, Eq. (4) results from substituting M as given in Eq. (1.17.3) or (1.17.5) into Eq. (1.17.1) and then dropping all nonlinear terms.

<u>Example</u>: In Fig. 1.18.1, B designates a rigid body that is attached by means of elastic supports to a space vehicle A which is moving in such a way that the angular velocity ${}^{N}\underline{\omega}^{A}$ of A in a newtonian reference frame N is given by

$$ {}^{N}\underline{\omega}^{A} = \alpha_1\underline{a}_1 + \alpha_2\underline{a}_2 + \alpha_3\underline{a}_3 \qquad \text{(a)} $$

where α_1, α_2, α_3 are constants and $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ form a dextral set of orthogonal unit vectors fixed in A. Point B* is the mass center of B, and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are unit vectors parallel to principal axes of inertia of B for B*, the associated moments of inertia having the values I_1, I_2, and I_3.

In preparation for the formulation of equations of motion of B, the first time-derivative, in N, of the angular momentum $\underline{H}$ of B relative to B* in N is to be determined, assuming that all rotational motions of B in A are slow, small motions. The orientation of B in A is to be described in terms of body-three-vector angles θ_1, θ_2, and θ_3, all of which vanish when $\underline{a}_i = \underline{b}_i$ (i = 1,2,3).

The angular velocity ${}^{N}\underline{\omega}^{B}$ of B in N can be expressed as

$$ {}^{N}\underline{\omega}^{B} \underset{(1.16.1)}{=} {}^{N}\underline{\omega}^{A} + {}^{A}\underline{\omega}^{B} \qquad \text{(b)} $$

Referring to Eqs. (1.2.5) and (1.8.12), one can write

$$ {}^{N}\underline{\omega}^{A} \underset{(a)}{=} (\alpha_1 - \alpha_2\theta_3 + \alpha_3\theta_2)\underline{b}_1 $$

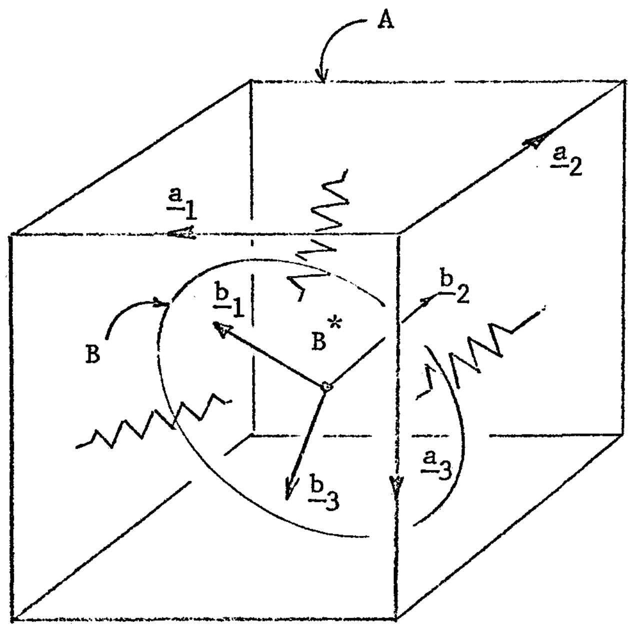

A
$\underline{a}_2$
$\underline{a}_1$
$\underline{b}_1$
$\underline{b}_2$
B
B^*
$\underline{a}_3$
$\underline{b}_3$

Figure 1.18.1

$$+ (\alpha_2 - \alpha_3\theta_1 + \alpha_1\theta_3)\underline{b}_2$$

$$+ (\alpha_3 - \alpha_1\theta_2 + \alpha_2\theta_1)\underline{b}_3 \qquad \text{(c)}$$

and, from Eq. (4)

$$^A\underline{\omega}^B = \dot{\theta}_1\underline{b}_1 + \dot{\theta}_2\underline{b}_2 + \dot{\theta}_3\underline{b}_3 \qquad \text{(d)}$$

Hence

$$^N\underline{\omega}^B \underset{(b,c,d)}{=} (\alpha_1 - \alpha_2\theta_3 + \alpha_3\theta_2 + \dot{\theta}_1)\underline{b}_1 + \cdots \qquad \text{(e)}$$

and

$$\underline{H} \underset{(e)}{=} I_1(\alpha_1 - \alpha_2\theta_3 + \alpha_3\theta_2 + \dot{\theta}_1)\underline{b}_1 + \cdots \qquad \text{(f)}$$

To evaluate the first time-derivative of $\underline{H}$ in N, it is convenient to use the relationship

$$\frac{^N d\underline{H}}{dt} \underset{(1.11.8)}{=} \frac{^B d\underline{H}}{dt} + {^N\underline{\omega}^B} \times \underline{H} \qquad \text{(g)}$$

with

$$\frac{^B d\underline{H}}{dt} \underset{(f)}{=} I_1(-\alpha_2\dot{\theta}_3 + \alpha_3\dot{\theta}_2 + \ddot{\theta}_1)\underline{b}_1 + \cdots \qquad \text{(h)}$$

and

$$^N\underline{\omega}^B \times \underline{H} \underset{(e,f)}{=} (I_3-I_2)(\alpha_2-\alpha_3\theta_1+\alpha_1\theta_3+\dot{\theta}_2)(\alpha_3-\alpha_1\theta_2+\alpha_2\theta_1+\dot{\theta}_3)\underline{b}_1 + \cdots \qquad \text{(i)}$$

where, however, all nonlinear terms are to be dropped prior to substituting from Eq. (i) into Eq. (g). Thus one finds that

$$\frac{{}^{N}d\underline{H}}{dt} = \left\{ I_1\ddot{\theta}_1 + (I_1 - I_2 + I_3)\alpha_3\dot{\theta}_2 - (I_1 + I_2 - I_3)\alpha_2\dot{\theta}_3 \right.$$

$$\left. + (I_3 - I_2)\left[\alpha_2\alpha_3 + \left(\alpha_2^2 - \alpha_3^2\right)\theta_1 - \alpha_1\alpha_2\theta_2 + \alpha_1\alpha_3\theta_3\right]\right\} \underline{b}_1 + \cdots$$

1.19 Instantaneous axis

At an instant at which the angular velocity $\underline{\omega}$ of a rigid body B in a reference frame A (see Sec. 1.11) is equal to zero, the velocities of all points of B in A are equal to each other. Whenever $\underline{\omega}$ is not equal to zero, there exist infinitely many points of B whose velocity in A is parallel to $\underline{\omega}$ or equal to zero. These points all have the same velocity $\underline{v}^*$ in A and they form a straight line parallel to $\underline{\omega}$ and called the instantaneous axis of B in A. The magnitude of $\underline{v}^*$ is smaller than the magnitude of the velocity in A of any point of B not lying on the instantaneous axis.

If $\underline{v}^Q$ is the velocity in A of an arbitrarily selected base-point Q of B, and P* is a point of the instantaneous axis, then the position vector $\underline{r}^*$ of P* relative to Q can be expressed as

$$\underline{r}^* = \frac{\underline{\omega} \times \underline{v}^Q}{\underline{\omega}^2} + \mu^*\underline{\omega} \tag{1}$$

where μ^* depends on the choice of P*; and $\underline{v}^*$ is given by

$$\underline{v}^* = \frac{\underline{\omega} \cdot \underline{v}^Q}{\underline{\omega}^2} \; \underline{\omega} \tag{2}$$

Derivation: In Fig. 1.19.1, both P and Q are arbitrarily selected points of B, $\underline{p}$ and $\underline{q}$ are their respective position vectors relative to a point O that is fixed in A, and $\underline{r}$ is the position vector of P relative to Q. Hence

$$\underline{p} = \underline{q} + \underline{r} \tag{a}$$

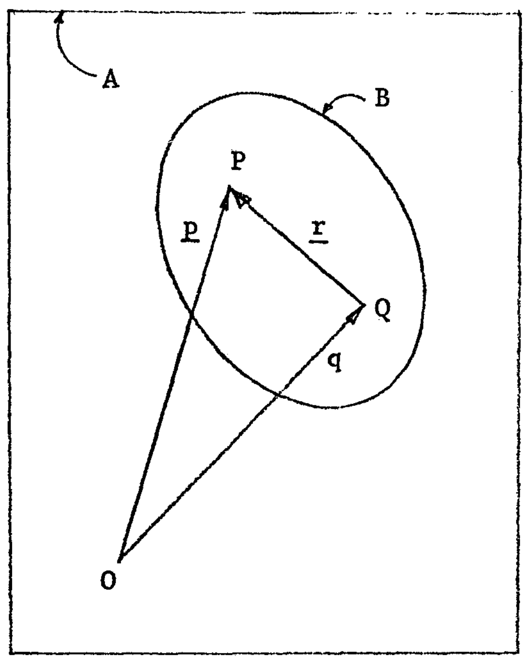


Figure 1.19.1

and

$$\frac{{}^A d\underline{p}}{dt} \underset{(a)}{=} \frac{{}^A d\underline{q}}{dt} + \frac{{}^A d\underline{r}}{dt} \underset{(1.11.9)}{=} \frac{{}^A d\underline{q}}{dt} + \underline{\omega} \times \underline{r}$$

or, since the velocities $\underline{v}^P$ and $\underline{v}^Q$ of P and Q in A are equal to ${}^A d\underline{p}/dt$ and ${}^A d\underline{q}/dt$,

$$\underline{v}^P = \underline{v}^Q + \underline{\omega} \times \underline{r} \tag{b}$$

If $\underline{\omega} \neq 0$, the vector $\underline{r}$ can always be expressed as the sum of a vector, say $\underline{s}$, that is perpendicular to $\underline{\omega}$, and the vector $\mu\underline{\omega}$, where μ is a certain scalar; that is

$$\underline{r} = \underline{s} + \mu\underline{\omega} \tag{c}$$

with

$$\underline{\omega} \cdot \underline{s} = 0$$

Consequently, $\underline{v}^P$ can be expressed as

$$\underline{v}^P \underset{(b,c)}{=} \underline{v}^Q + \underline{\omega} \times \underline{s} \tag{e}$$

and

$$\underline{\omega} \times \underline{v}^P \underset{(e)}{=} \underline{\omega} \times \underline{v}^Q + \underline{\omega} \cdot \underline{s}\underline{\omega} - \underline{\omega}^2 \underline{s}$$

$$\underset{(d)}{=} \underline{\omega} \times \underline{v}^Q - \underline{\omega s} \tag{f}$$

If P is now taken to be a point P^* whose velocity $\underline{v}^*$ in A is parallel to $\underline{\omega}$, and the associated values of $\underline{r}$, $\underline{s}$, and μ are called $\underline{r}^*$, $\underline{s}^*$, and μ^*, then

$$0 \underset{(f)}{=} \underline{\omega} \times \underline{v}^Q - \underline{\omega}^2 \underline{s}^* \qquad \text{(f)}$$

$$\underline{r}^* \underset{(c)}{=} \underline{s}^* + \mu^* \underline{\omega} \underset{(g)}{=} \frac{\underline{\omega} \times \underline{v}^Q}{\underline{\omega}^2} + \mu^* \underline{\omega}$$

in agreement with Eq. (1), and

$$\begin{aligned}
\underline{v}^* &\underset{(e)}{=} \underline{v}^Q + \underline{\omega} \times \underline{s}^* \\
&= \underline{v}^Q + \frac{\underline{\omega} \times (\underline{\omega} \times \underline{v}^Q)}{\underline{\omega}^2} \\
&= \frac{\underline{\omega}^2 \underline{v}^Q + \underline{\omega} \cdot \underline{v}^Q \underline{\omega} - \underline{\omega}^2 \underline{v}^Q}{\underline{\omega}^2} = \frac{\underline{\omega} \cdot \underline{v}^Q}{\underline{\omega}^2} \, \underline{\omega}
\end{aligned}$$

in agreement with Eq. (2).

<u>Example</u>: In Fig. 1.19.2, B represents a slowly spinning cylindrical satellite whose mass center B^* moves on a circular orbit of radius R fixed in a reference frame A. Throughout this motion the symmetry axis of B is constrained to remain tangent to the circle while B rotates about this axis at a constant rate such that a plane fixed in B and passing through the axis becomes

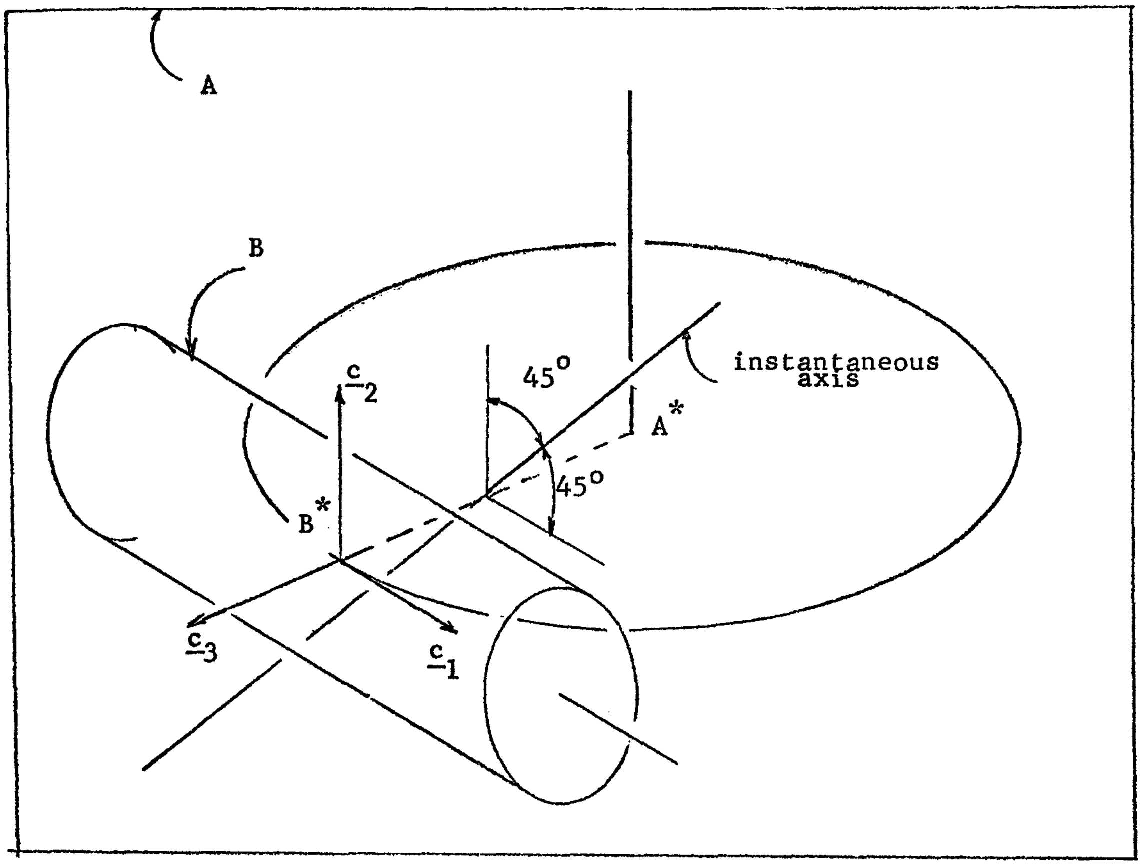

A
B
$\underline{c}_2$
45°
instantaneous
axis
A^*
45°
B^*
$\underline{c}_3$
$\underline{c}_1$

Figure 1.19.2

parallel to the orbit plane twice during each orbital revolution of B*. The instantaneous axis of B in A is to be located for a typical instant during the motion.

Letting A* be the center to the circle on which B* moves, and designating as C a reference in which the normal to the circle at A* and the line joining A* to B* are both fixed, one can express the angular velocity of B in A as

$$\underline{\omega} \underset{(1.12.1)}{=} {}^{A}\underline{\omega}^{C} + {}^{C}\underline{\omega}^{B} \tag{a}$$

Furthermore, if Ω denotes the rate at which the line joining A* to B* rotates in A, then

$${}^{A}\underline{\omega}^{C} \underset{(1.11.5)}{=} \Omega\, \underline{c}_2 \tag{b}$$

and

$${}^{C}\underline{\omega}^{B} \underset{(1.11.5)}{=} \Omega\, \underline{c}_1 \tag{c}$$

where $\underline{c}_1$ and $\underline{c}_2$ are unit vectors directed as in Fig. 1.19.2. Hence

$$\underline{\omega} \underset{(a-c)}{=} \Omega(\underline{c}_1 + \underline{c}_2) \tag{d}$$

The velocity $\underline{v}^{B*}$ of B* in A is given by

$$\underline{v}^{B*} = R\Omega\underline{c}_1 \tag{e}$$

Consequently, if P^* is a point on the instantaneous axis of B in A, then the position vector $\underline{r}^*$ of P^* relative to B^* is given by

$$\underline{r}^* \underset{(1)}{=} \frac{\Omega(\underline{c}_1 + \underline{c}_2) \times (R\Omega\underline{c}_1)}{2\Omega^2} + \mu^*\Omega(\underline{c}_1 + \underline{c}_2)$$

$$= -\frac{R}{2}\underline{c}_3 + \mu^*\Omega(\underline{c}_1 + \underline{c}_2)$$

Hence the instantaneous axis of B in A passes through the midpoint of the line joining A^* to B^*, is perpendicular to $\underline{c}_3$, and makes a forty-five degree angle with each of $\underline{c}_1$ and $\underline{c}_2$, as indicated in Fig. 1.19.2.

1.20 Angular acceleration

The angular acceleration $\underline{\alpha}$ of a rigid body B in a reference frame A is defined as the first time-derivative in A of the angular velocity $\underline{\omega}$ of B in A (see Sec. 1.11):

$$\underline{\alpha} \triangleq \frac{{}^{A}d\underline{\omega}}{dt} \tag{1}$$

Frequently it is convenient to resolve both $\underline{\omega}$ and $\underline{\alpha}$ into components parallel to unit vectors fixed in a reference frame C, that is, to express $\underline{\omega}$ and $\underline{\alpha}$ as

$$\underline{\omega} = {}^{C}\omega_1\underline{c}_1 + {}^{C}\omega_2\underline{c}_2 + {}^{C}\omega_3\underline{c}_3 \tag{2}$$

and

$$\underline{\alpha} = {}^{C}\alpha_1\underline{c}_1 + {}^{C}\alpha_2\underline{c}_2 + {}^{C}\alpha_3\underline{c}_3 \tag{3}$$

where $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ is a dextral set of orthogonal unit vectors. When this is done,

$${}^{C}\alpha_i = {}^{C}\dot{\omega}_i + \underline{\Omega} \times \underline{\omega} \cdot \underline{c}_i \quad (i = 1,2,3) \tag{4}$$

where $\underline{\Omega}$ is the angular velocity of C in A. In other words, depending on the motion of C in A, ${}^{C}\alpha_i$ may, or may not, be equal to ${}^{C}\dot{\omega}_i$.

<u>Derivations</u>: Using Eq. (1.11.8), one can express $\underline{\alpha}$ as

$$\underset{(1)}{\underline{\alpha} =} \frac{{}^{C}d\underline{\omega}}{dt} + \underline{\Omega} \times \underline{\omega}$$

$$\underset{(2)}{=} {}^{C}\dot{\omega}_i \underline{c}_i + {}^{C}\dot{\omega}_2 \underline{c}_2 + {}^{C}\dot{\omega}_3 \underline{c}_3 + \underline{\Omega} \times \underline{\omega}$$

Consequently, when $\underline{\alpha}$ is expressed as in Eq. (3), then

$${}^{C}\alpha_1 \underline{c}_1 + {}^{C}\alpha_2 \underline{c}_2 + \alpha_3 c_3 = {}^{C}\dot{\omega}_i \underline{c}_i + {}^{C}\dot{\omega}_2 \underline{c}_2 + {}^{C}\dot{\omega}_3 \underline{c}_3 + \underline{\Omega} \times \underline{\omega}$$

and dot-multiplication with $\underline{c}_i$ $(i = 1,2,3)$ gives

$${}^{C}\alpha_i = {}^{C}\dot{\omega}_i + \underline{\Omega} \times \underline{\omega} \cdot \underline{c}_i \quad (i = 1,2,3)$$

Example: Fig. 1.20.1 depicts the system previously considered in the Example in Sec. 1.12. In addition to the unit vectors used previously, orthogonal unit vectors $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ are shown, and a reference frame C, in which these are fixed, is indicated. Considering only motions such that $\dot{\phi}$ and $\dot{\psi}$, as well as θ, remain constant, the quantities ${}^{A}\alpha_i$, ${}^{B}\alpha_i$ and ${}^{C}\alpha_i$ $(i = 1,2,3)$ are to be determined, these being defined as

$${}^{A}\alpha_i \triangleq \underline{\alpha} \cdot \underline{a}_i, \; {}^{B}\alpha_i \triangleq \underline{\alpha} \cdot b_i, \; {}^{C}\alpha_i \triangleq \underline{\alpha} \cdot c_i \quad (i = 1,2,3) \qquad \text{(a)}$$

where $\underline{\alpha}$ is the angular acceleration of B in A.

The angular velocity $\underline{\omega}$ of B in A can be expressed as

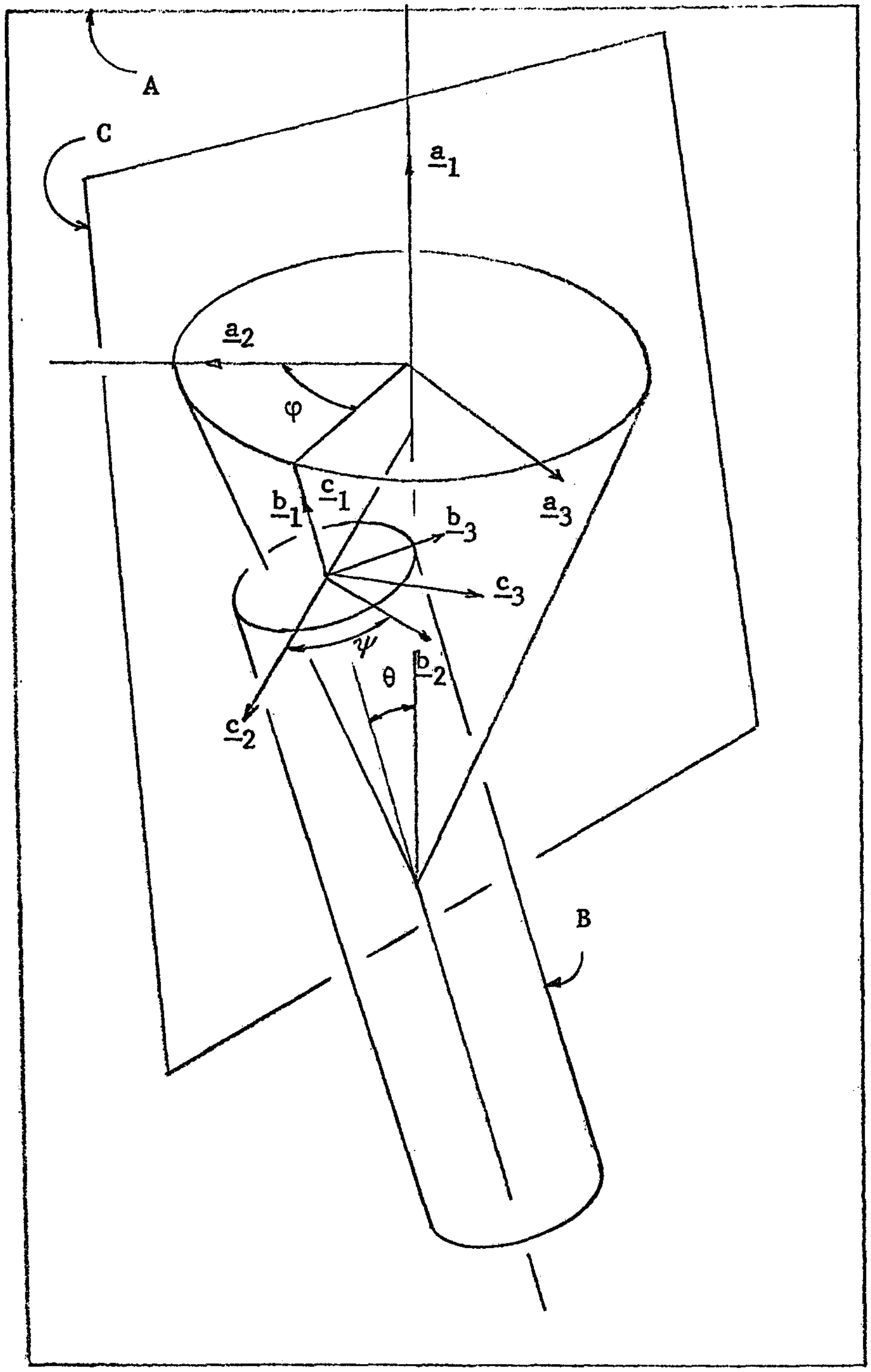

A
C
$\underline{a}_1$
$\underline{a}_2$
φ
$\underline{a}_3$
$\underline{b}_1$
$\underline{c}_1$
$\underline{b}_3$
$\underline{c}_3$
ψ
$\underline{b}_2$
θ
$\underline{c}_2$
B

Figure 1.20.1

$$\underline{\omega} = (\dot{\psi}\ c\theta + \dot{\phi})\underline{a}_1 + \dot{\psi}\ s\theta\ c\phi\ \underline{a}_2 + \dot{\psi}\ s\theta\ s\phi\ \underline{a}_3 \tag{b}$$

or as

$$\underline{\omega} = (\dot{\psi} + \dot{\phi}\ c\theta)\underline{b}_1 - \dot{\phi}\ s\theta\ c\psi\ \underline{b}_2 + \dot{\phi}\ s\theta\ s\psi\ \underline{b}_3 \tag{c}$$

or as

$$\underline{\omega} = (\dot{\psi} + \dot{\phi}\ c\theta)\underline{c}_1 - \dot{\phi}\ s\theta\ \underline{c}_2 \tag{d}$$

Using Eq. (4), with C replaced by A, and hence $\underline{\Omega} = 0$, one obtains by reference to Eq. (b)

$$^A\alpha_1 = \frac{d}{dt}(\dot{\psi}\ c\theta + \dot{\phi}) = 0 \tag{c}$$

$$^A\alpha_2 = \frac{d}{dt}(\dot{\psi}\ s\theta + c\phi) = -\dot{\psi}\dot{\phi}\ s\theta\ s\phi \tag{f}$$

$$^A\alpha_3 = \frac{d}{dt}(\dot{\psi}\ s\theta\ s\phi) = \dot{\psi}\dot{\phi}\ s\theta\ c\phi \tag{g}$$

Similarly, with C replaced by B in Eq. (4), so that $\underline{\Omega} = \underline{\omega}$, Eq. (c) permits one to write

$$^B\alpha_1 = \frac{d}{dt}(\dot{\psi} + \dot{\phi}\ c\theta) = 0 \tag{h}$$

$$^B\alpha_2 = \frac{d}{dt}(-\dot{\phi}\ s\theta\ c\psi) = \dot{\phi}\dot{\psi}\ s\theta\ s\psi \tag{i}$$

$$^B\alpha_3 = \frac{d}{dt}(\dot{\phi}\ s\theta\ c\psi) = \dot{\phi}\dot{\psi}\ s\theta\ c\psi \tag{j}$$

Finally, with

$$\underline{\Omega} = {}^A\underline{\omega}^C = \dot{\phi}(c\theta\ \underline{c}_1 - s\theta\ \underline{c}_2) \tag{k}$$

so that

$$\underline{\Omega} \times \underset{(k,j)}{\underline{\omega}} = \dot{\phi}\dot{\psi}\, s\theta\, \underline{c}_3 \qquad (\ell)$$

it follows from Eq. (4) together with Eqs. (d) and (ℓ) that

$$^{C}\alpha_1 = \frac{d}{dt}(\dot{\psi} + \dot{\phi}\, c\theta) + \dot{\phi}\dot{\psi}\, s\theta\, \underline{c}_3 \cdot \underline{c}_1 = 0$$

$$^{C}\alpha_2 = \frac{d}{dt}(-\dot{\phi}\, s\theta) + \dot{\phi}\dot{\psi}\, s\theta\, \underline{c}_3 \cdot c_2 = 0$$

and

$$^{C}\alpha_3 = \dot{\phi}\dot{\psi}\, s\theta$$

Thus, with the exception of $^{C}\alpha_3$, every one of the quantities defined in Eqs. (a) is equal to the time derivative of the corresponding angular velocity measure number.